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MAXWELL'S EQUATIONS AND ATOMIC DYNAMICS

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The electromagnetic equations have generally been considered as determining the nature of the field when the motion of the charge is known. If, however, we were in a position to represent the electromagnetic forces on the surface and within the charge as given functions of the position and velocities of the charge then Maxwell's equations could be regarded as the equation of motion of the electric particle since these electromagnetic equations hold not only for free space but also within the boundary of the electric charge. In this paper, a certain form is taken as representing the electric and magnetic forces within the charge as given functions of the velocities of the moving body and of its position. This solution satisfies the first of Maxwell's equations identically and gives the second set as the equations of motion of the charge for which this solution is valid. Among the solutions of the differential equations of this motion, there are a certain set for which the energy integral is constant—this set of paths, which we shall term non-radiating, being determined by the Sommerfeld quantum conditions. Using the results obtained by this analysis, we have been able, following DeBroglies' ideas, to obtain the Bohr frequency condition as a consequence of these ideas.

The electromagnetic equations for any coördinate system in the abbreviated tensor form can be written:

$$\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{ik}}{\partial x^j} + \frac{\partial F_{kj}}{\partial x^i} = 0 \qquad (1)$$

and

$$(F^{i\alpha})_{\alpha} = J^i \qquad (2)$$

where the Greek indices repeated will be understood to be summed from 1 to 4, x^1, x^2 and x^3 representing the spatial and x^4 the time coördinate, where F_{ij} represents the components of the electric and magnetic intensities with their proper signs and J^i the current density (Eddington—

"Math. Theory of Relativity," Cambridge Univ. Press, pp. 171 to 175).

The operation of covariant differentiation, indicated above, acting on a field force F^{ij} is defined in general coördinates by the relation

$$(F^{i\alpha})_\alpha = \frac{\partial F^{i\alpha}}{\partial x^\alpha} + \Gamma^i_{\beta\alpha}F^{\beta\alpha} + \Gamma^\alpha_{\beta\alpha}F^{\beta i}$$

where

$$F^{ij} = g^{i\alpha}g^{i\beta}F_{\alpha\beta}$$

$$\Gamma^i_{jk} = {}^1/_2 g^{i\alpha}\left(\frac{\partial g_{j\alpha}}{\partial xk} + \frac{\partial g_{k\alpha}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\alpha}\right)$$

and

$$g^{i\alpha}g_{j\alpha} = \delta^i_j \begin{matrix} = 0 & i \neq j \\ = 1 & i = j \end{matrix}$$

the g's being the coefficients in the energy tensor H

$$H = g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \tag{3}$$

where H is the energy of a charge of mass m and charge e which moves in the space according to the Coulomb Law neglecting the reaction of the field of the moving particle on itself. This energy function H corresponds to the energy of the electron in the method used by Bohr and others in the treatment of quantum dynamics.

The current density function J^i has the form $J^i = \rho\, dx^i/ds$, while the continuity equation gives the condition $J^\alpha_\alpha = (\rho\, dx^\alpha/ds)_\alpha$.

Since the only case we can solve completely is when the equations of motion can be expressed in such a form that the variables are separable, we shall limit the discussion to this case, then

$$g_{i\alpha}\, dx^i/ds = Q_i(x^i)$$

where $Q_i(x^i)$ is a function of the coördinate x^i only.

We shall take as the value of the electric and magnetic forces within and on the boundary of the moving charge $F_{ij} = Q_iQ_jI_{ij}$, where the quantities I_{ij} have the skew-symmetric property that $I_{ij} = -I_{ji}$ and $I_{ii} = 0$ (not summed) the equations of motion being determined in such a manner that Maxwell's equations are satisfied for this form of the electromagnetic forces.

The first set of Maxwell's equations (1) are satisfied for this form of the forces and the second set (2), namely

$$(Q^iQ^\alpha)_\alpha . I_{i\alpha} = \rho Q^i \tag{4}$$

$$Q^i = g^{i\alpha}Q_\alpha$$

become the equations of motion of the moving charge under the influence of its own electromagnetic field and the external field subject to the two conditions:

(I) That the relation $J_\alpha^\alpha = 0$ is satisfied.

(II) That the differential equations (4) are of such a form that the momenta are functions of the corresponding coördinates only.

If we now limit ourselves thus further to the case where $Q_i^i = 0$ (not summed) and make a contact transformation on the momenta only such that the energy and action integral are invariants for this transformation, then equation (4) takes the form $Q^\alpha Q_\alpha^i = GQ^\beta Q_\beta Q^i$, where G is a scalar, an equation which is identical with the equation of the geodesics in a Riemann space where the vector appearing there is now the momentum corresponding to the motion, the first integral of the equations of motion being of the form

$$e^{\int G\, g_{\alpha\beta} \frac{dx^\alpha}{ds} dx^\beta} \qquad g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \text{constant.} \qquad (5)$$

When, however, the paths are periodic, the above integral must satisfy the condition

$$\int G\, g_{\alpha\beta} \frac{dx^\alpha}{ds}\, dx^\beta = 2\pi n i \qquad \begin{cases} n = \text{integer} \\ i = \sqrt{-1} \end{cases} \qquad (6)$$

the integral being taken over a complete cycle. In the case where the variables are separable, this condition reduces to the Sommerfeld quantum conditions with the addition, however, of the quantum integral with respect to the time coördinate. This extra time coördinate has been used by the writer to give an account of the multiplet structure of lines (*Phil. Mag.*, June, 1925) and the relation of the multiplet and zeeman structures (*J. Frank. Inst.*, Jan., 1926) giving formula for these phenomena which agrees well with experiment. The fourth quantum condition has also been used by P. A. M. Dirac (*Proc. Royal Soc.*, June, 1926) to account for the Compton effect on the wave theory.

From (5) we see that the energy is constant or periodic along a path for which the conditions (6) are true, so that the path of a non-radiating electron obeying Maxwell's equations must satisfy the equations of motion of a Riemannian geodesic (with arbitrary gauge factor) in a four dimensional space subject to the condition

$$\int p_i\, dq^i = nh$$

h = constant; p_i being the momentum corresponding to the coördinate q^i where the equations of motion are in such a form that the variables are separable.

In moving from one quantized path with energy H' and momenta

p_i' to another with energy H'' and momenta p_i'', the electron will emit radiation of energy $H' - H''$, the direction of the electromagnetic wave being determined by the electromagnetic momentum vector. The radiation wave, moreover, will also be determined by the electromagnetic potentials k_i, found as solutions of Maxwell's equations for free space, which have the form at a great distance from the charge system of

$$K_i = A_i e^{l_1 x^1 + l_2 x^2 + l_3 x^3 - l_4 x^4} \tag{7}$$

where the directions cosines in the four dimensional space, l_1, l_2, l_3 and l_4 will be proportional to the components of the electromagnetic momentum vector M_i of the wave. Since this momentum vector is equal to the energy flux divided by one-half the square of the velocity of propagation, we have

$$l_i = k' M_i = k\left(\frac{p_i'^2}{g_{ii}'} - \frac{p_i''^2}{g_{ii}''}\right). \tag{8}$$

From this conclusion we see that in order to have interference the direction of the two waves in the four-dimensional space must be identical, the nature of the interference total or partial depending on the relative place of the two waves.

If we write (7) in the familiar form

$$k_i = A_i\, e^{2\pi i \frac{\nu}{V}(\cos\alpha\, x + \cos\beta\, y + \cos\gamma\, z - Vt)}$$

where ν = frequency of the radiation
and V = velocity of propagation
we are lead to the following theorem:

THEOREM. *The frequency of ν of an electromagnetic wave is equal to the direction cosine between the time axis and the wave normal.*

We shall now apply this method for the determination of the frequency of an electromagnetic wave to the case of an electron revolving about a positive charge E.

The energy H of a non-radiating electron moving in a plane orbit is

$$-H = \left(1 - \frac{2eE}{r}\right)\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 - \left(1 - \frac{2eE}{r}\right)\left(\frac{dt}{ds}\right)^2$$

$$= g_{\alpha\alpha}\left(\frac{dx^\alpha}{ds}\right)^2$$

which determines the g's.

The equations of motion of the electron in the special case of a circular orbit, i.e., $\frac{d'r}{ds^2} = \frac{dr}{ds} = 0$ are:

$$r\left(1-\frac{2eE}{r}\right)\left(\frac{d\theta}{ds}\right)^2-\frac{eE}{r}\left(1-\frac{2eE}{r}\right)\left(\frac{dt}{ds}\right)^2=0$$

$$\frac{d^2\theta}{ds^2}+\frac{d}{r}\left(\frac{dr}{ds}\right)\left(\frac{d\theta}{ds}\right)=G\,g_{\alpha\alpha}\left(\frac{dx^\alpha}{ds}\right)^2\frac{d\theta}{ds}$$

and

$$\frac{d^2t}{ds^2}+\frac{d}{dr}\left(\log\left(1-\frac{2eE}{r}\right)\right)\left(\frac{dt}{ds}\right)\left(\frac{dr}{ds}\right)=G\,g_{\alpha\alpha}\left(\frac{dx^\alpha}{ds}\right)^2\frac{dt}{ds}$$

whose solutions are,

$$p_3=r\frac{d\theta}{ds}=P_3\,e^{\int G\,g_{\alpha\alpha}\left(\frac{dx^\alpha}{ds}\right)dx^\alpha}$$

$$p_4=\left(1-\frac{2eE}{r}\right)\frac{dt}{ds}=P_4\,e^{\int G\,g_{\alpha\alpha}\left(\frac{dx^\alpha}{ds}\right)dx^\alpha}$$

and

$$g_{\alpha\alpha}\left(\frac{dx^\alpha}{ds}\right)^2=H_o^2\,e^{2\int G\,g_{\alpha\alpha}\left(\frac{dx^\alpha}{ds}\right)dx^\alpha}$$

where P_3, P_4 and H_o^2 are constants of integration.

As

$$H=g_{\alpha\alpha}\left(\frac{dx^\alpha}{ds}\right)^2$$

$$=H_o^2\,e^{2\int G\,g_{\alpha\alpha}\left(\frac{dx^\alpha}{ds}\right)dx^\alpha}$$

we find after simplification and evaluation of the constants of integration that

$$M_4=k\left[\frac{p_4'^2}{g_{44}'}-\frac{p_4''^2}{g_{44}''}\right]=k(H'-H'')$$

but since the frequency of the electromagnetic wave is proportional to the time component of the wave momentum M_4, then the frequency ν of the light emitted by the radiating electron in moving from one stationary state to another is given by the relation

$$\nu=h(H'-H'')\left(h=\frac{\text{universal}}{\text{constant}}\right)$$

the Bohr frequency condition.

A detailed discussion of these results appears in the December issue of the *Journal of the Franklin Institute.*

THE SPECTRUM OF KRYPTON IN THE EXTREME ULTRA-VIOLET

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Communicated October 30, 1926

As a preliminary step toward the analysis of the spectrum of krypton the writer has recently made an investigation of the extreme ultra-violet spectrum of that element. The vacuum spectrograph and glass concave grating employed have been described elsewhere.[1] The krypton gas was obtained from the Linde Ice-Machine Company in Münich and was found free from impurities except for traces of argon and xenon, the resonance lines of which appear on the plates.

The wave-lengths of the stronger lines were determined from several plates using known hydrogen, oxygen and nitrogen lines as standards. In the following table are given preliminary values of these wave-lengths together with those of the resonance lines of argon, krypton and xenon, as recently announced independently by G. Hertz.[2]

	INT.	λ VAC.	λ VAC. (HERTZ)
	12	1235.87	1235.8
	10	1164.90	1164.9
	2	1074.52	
Krypton	2	1067.11	
	3	1051.97	
	2	1041.38	
	4	1003.46	
Argon	0	1066.62	1066.8
	4	1048.21	1048.3
Xenon	8	1469.67	1469.5
	4	1295.65	1295.7

Several interesting intensity relations may be noted. In the first place it is seen that whereas in argon and neon the short wave-length resonance line has been found to be the stronger member of the pair, in krypton and xenon it is the long wave-length member which is the more intense.

Further, Lyman and Saunders[3] have noted a marked reduction in the intensity of the short wave-length member of the neon pair when neon is present as an impurity in helium. The intensities given above for xenon indicate that the same effect may occur, although to a less degree, when xenon is present as an impurity in krypton. This is uncertain, however, no data being at hand as to the relative intensities of these lines in pure xenon, but comparison with the other rare gases indicates that these lines would differ less in intensity than is shown by the above data.

Finally it may be noted that when argon is present as an impurity in

krypton the long wave-length member of the argon pair suffers a reduction in intensity. This fact is particularly interesting inasmuch as Lyman and Saunders have found that when argon is present as an impurity in neon, there is no such change in the relative intensity of the argon lines.

Calculation yields for the resonance pair of krypton a frequency difference of 4930, which is found repeated in other parts of the spectrum, e.g.,

λ	$\Delta\nu$
8281.02	4930.1
5879.84	
8263.22	4930.0
5870.90	

These, and other known frequency differences, are furnishing a starting point for the analysis of the spectrum.

[1] T. Lyman, *Astroph. J.*, **60**, 1 (1924); R. W. Wood and T. Lyman, *Phil. Mag.*, **2** 7, 310 (1926).

[2] G. Hertz, *Naturwissenschaften*, **14**, 27, 648 (July 2, 1926).

[3] T. Lyman and F. A. Saunders, these PROCEEDINGS, **12**, 2 (1926).

SOME THERMIONIC EXPERIMENTS WITH A NEW SOURCE OF POSITIVE IONS

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It was found that fused mixtures of iron oxide containing about 1% of an oxide of an alkali or alkaline earth metal furnished a very convenient source of positive ions when used as the hot anode in a vacuum tube.[1] That the positive ions emitted were single atoms stripped of one electron and were very largely the ions of the alkali or alkaline earth metal which was introduced in the original mixture, was shown by a mass spectrograph analysis by Barton and Harnwell at Princeton.[2]

The ease of operation and reproducibility of the thermionic emission from these surfaces, both positive and negative, enabled experimental results to be obtained which throw considerable light on the mechanism of this emission. The finely powdered iron alkali oxide mixture was coated and fused onto a twisted platinum strip until the entire surface of the strip was covered with a uniform coating of the material.[3]

The coated strip was mounted as F_1 in a vacuum tube as shown in figure 1, and reduced in an atmosphere of hydrogen at a dull red temperature.

A tungsten filament F_2, enabled the collecting cylinder P_1 and the guard cylinder P_2 to be thoroughly degassed by electron bombardment. A current was passed through F_1 and the surface brought to the proper temperature as determined by an optical pyrometer.

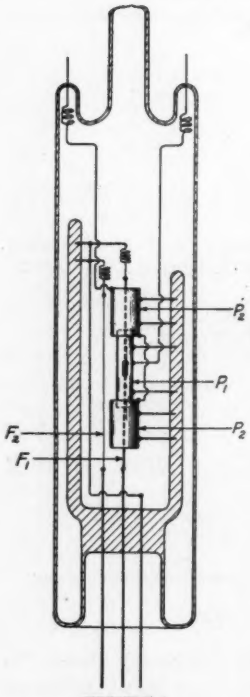

FIGURE 1

In every case, after a preliminary glowing of the coated strip, both the positive ion and the electron emission obeyed Richardson's equation $I = AT^{1/2}\epsilon^{-b/T}$, through a considerable range of temperature at a pressure of 10^{-6} mm. of Hg. In the equation I is the saturation current positive or negative, A and b constants, A representing the current emitted per unit area of the hot surface. The b constant will be expressed in terms of ϕ, the equivalent voltage of the work function for the vaporization of a positive ion or an electron, by the relation $b = \phi e/k$; where e is the unit electric charge and k the Boltzmann gas constant.

In figure 2, the $\log I_+ - {}^1/_2 \log T$ is plotted against the reciprocal of the absolute temperature, the usual method of applying Richardson's equation. ϕ_+ which is proportional to the slope of the line is 1.96 volts for Cs ions from an Fe, Al, Cs mixture. A similar result for K ions gave ϕ_+ as 2.10 volts.

In the case of a thoroughly reduced Fe-Ba mixture, the Ba ion emisson was quite constant at sufficiently high temperatures, where appreciable electron currents were also obtained. That is, at a given temperature, the hot surface was a constant source of positive Ba ions, when used as the anode, and of electrons when used as a cathode. The respective curves, figure 3, represent the positive Ba ion and electron emission from such a surface. ϕ_+ for the Ba ions is 2.12 volts for a temperature range of 920–1330°K.; where ϕ_- for the electrons is 3.49 volts for a temperature range of from 1136° to 1370°K. Similar results were also obtained for Sr and Ca.

Considerable significance may be attached to the relative values of ϕ_+ and ϕ_- in regard to the mechanism of escape of the positive ions from the surface.[4] In every case where it was possible to measure both ion and electron currents for the same surface, ϕ_- was found larger than ϕ_+. This is probably why the Ba atom is robbed of one of its outer electrons

on vaporizing from the surface. In other words, the electron affinity or attraction of the surface for an outer electron of the Ba atom is greater than the force tending to hold the electron to the Ba atom. However, $\phi-$ is not large enough to remove both outer electrons from the Ba atom on vaporization from the surface, as no doubly charged Ba ions were observed in the mass spectrograph studies. At the higher temperatures an appreciable vaporization of Fe also took place, but as neutral vapor since no Fe ions were observed. We may conclude then that on the vaporization of an element from a surface, the vaporization will be in the form of positive ions providing the electron work function $\phi-$ of the surface is greater than the potential tending to hold the outer or conduction electron to the atom about to be vaporized.

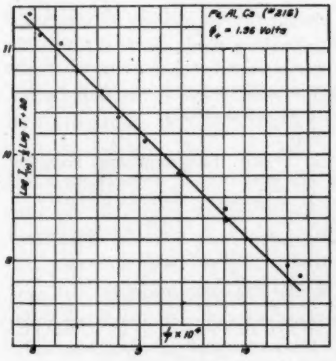

FIGURE 2

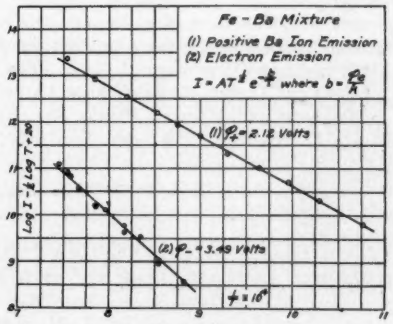
Fe - Ba Mixture
(1) Positive Ba Ion Emission
(2) Electron Emission
$I = AT^{\frac{1}{2}} e^{-\frac{b}{T}}$ where $b = \frac{\phi_2}{k}$
(1) ϕ_2 = 2.12 Volts
(2) ϕ_- = 3.49 Volts
Log I - ½ Log T - b
$\frac{1}{T} \times 10^4$

FIGURE 3

One would not expect that the surface when electrons are collected was identical to the surface when positive ions are collected. Some experimental evidence tends to indicate that this difference or change in the surface is relatively small in comparison to the entire surface. From the results of X-ray analysis which indicate that the alkali oxide is distributed between crystals,[5] and the fact that diffusion takes place largely along interfaces between crystals;[6] we conclude that the positive ion emission is largely an interface phenomena and is, therefore, confined to a small part of the total surface. Even with the collector positive with respect to the hot surface we can think of a maximum concentration of the alkali or alkaline earth metal being built up at the interfaces, any change of which must take place in the vaporization of neutral atoms. If an appreciable part of the surface were covered with the alkali or alkaline earth metal, much smaller values of $\phi-$ would result besides a relative large electron emission would be present at the lower temperatures, as shown by the work on thin films.[7]

While these experiments were concerned largely with a determination of ϕ_+, the equivalent voltage of the work function for the removal of a positive ion from the hot surface, other experiments with positive ions may be simplified by the use of this ion source.

In conclusion, the author wishes to express his appreciation for the interest shown in this work by the members of the Laboratory staff, and to Prof. K. T. Compton of Princeton for making the mass spectrograph analysis possible, and especially to Mr. R. A. Nelson for his careful work in the construction of the apparatus and assistance in the experimental work.

[1] *Science*, **62**, 269 (1925); *Physic. Rev.*, **25**, 892 (1925).

[2] *Physic. Rev.*, **27**, 739 (1926).

[3] *J. Phys. Chem.*, **30**, 525 (1926).

[4] For an exact determination of ϕ, a method similar to that used by Davisson and Germer (*Phys. Rev.*, **20**, 300 (1922); **24**, 666 (1924)) should be used. However, with the present method, a value of ϕ_- for pure tungsten was obtained as 4.25 volts, where the value determined by Davisson and Germer with the same equation was 4.78 volts.

[5] Wychoff and Crittenden, *J. Amer. Chem. Soc.*, **47**, 2866 (1925).

[6] Geiss and Liempt, *Z. Metallkunde*, **16**, 317 (1924).

[7] Davisson and Pidgeon, *Physic. Rev.*, **15**, 553 (1920); Langmuir and Kingdon, *Science*, **57**, 58 (1923); *Proc. Roy. Soc.*, **A107**, 61 (1925).

THE DIELECTRIC CONSTANT AND DIAMAGNETISM OF HYDROGEN AND HELIUM IN THE NEW QUANTUM MECHANICS

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The new Heisenberg-Schrödinger quantum mechanics enables one to calculate exactly the diamagnetic and dielectric susceptibilities of hydrogenic atoms. We shall see that whenever the quantum numbers are small the numerical values obtained for these susceptibilities are considerably different from those of the old quantum theory, and usually in more convincing agreement with experiment.

Dielectric Constant of Atomic Hydrogen in the Normal State.—After application of a field F the energy of a state n is, in general, of the form

$$W_n = W_{0n} + a_1F + {}^1/_2a_2F^2 + \ldots, \qquad (1)$$

where W_{0n} is the energy in the absence of the field. Waller[1] and Epstein[2] have independently recently shown that in a strong electric field F the coefficients a_1 and a_2 have the values

$$a_1 = \frac{3h^2n(m_1 - m_2)}{8\pi^2\mu eZ}, \qquad a_2 = -\frac{n^4h^6}{512\pi^6Z^4e^6\mu^3}[17n^2 - 3(m_1 - m_2)^2 - 9m_3^2 + 19], \quad (2)$$

where Ze is the nuclear charge, μ is the mass of an electron, and m_1, m_2, m_3 are quantum numbers which can assume all values from 0 to $n - 1$. The principal quantum number n is $m_1 + m_2 + m_3 + 1$. The formula (2) differs from that of the old quantum theory only in the addition of the constant $+19$ in the bracketed factor of a_2, and in the fact that the equatorial quantum number m_3 is uniformly one unit lower than previously.

Let us suppose that the hydrogen atoms are all in their lowest or normal state. Then $n = 1$, and the m's all vanish, so that by (2) $a_1 = 0$. There is then no first order Stark effect, and if there are N atoms per cc., the dielectric constant ϵ is given by

$$\epsilon - 1 = -4\pi Na_2 = 9Nh^6/32\pi^5Z^4e^6\mu^3, \quad (3)$$

as it is easily shown that when $a_1 = 0$ the polarization induced in an individual atom is $-a_2E$, equally well in the new or old mechanics. This fact has already been utilized by Waller[1] in an interesting calculation of the polarization of the inner orbit by the outer electron in excited helium, and the resulting contribution to the "quantum defect" in the Rydberg formula. If $Z = 1$, as in hydrogen, formula (3) gives $\epsilon = 1.000229$ at 0°C. and atmospheric pressure. The old quantum theory would give $\epsilon = 1.000051$, as a_2 is 4.5 times greater in the new than in the old theory when $n = 1$. Adequate experimental evidence on the dielectric constant of monatomic hydrogen is unfortunately not at present available due to the difficulty of dissociating hydrogen gas into the atomic form.*

At this point the question naturally arises whether the dielectric constant in weak fields may not differ from the value we have given because the relativity corrections are neglected in (2), whereas actually in very weak fields in ordinary mechanics the relativity precession may be larger than the change in frequency produced by the electric field. In the old quantum theory such a suggestion has been made by Miss Aylesworth.[3] However, in the new quantum mechanics the analog of the relativity precession frequency is the spectroscopic frequency emitted or absorbed when the azimuthal quantum number changes and the principal quantum number is unaltered. Such frequencies are not involved in the normal state of hydrogenic atoms, for this is a singlet level characterized by but one value of the azimuthal quantum number, and is not even split into component levels by an electric field. Therefore in the normal state there is no degeneracy introduced by omitting relativity corrections and external fields, and we do not need to consider the secular perturbations whose technique has been developed by Born, Heisenberg and Jordan,[4] and by

Schrödinger,[5] and which are the analog in the new mechanics of the "long-period terms" of the old theory. Born, Heisenberg and Jordan's determinantal equation (25) (loc. cit.,[4] p. 589) is easily seen to reduce to a single element in the normal state and to be precisely the same as in the absence of the field. Thus there is no secular change in energy, and the quadratic Stark effect in the normal state arises entirely from amplitudes associated with transitions to the higher quantum states $n = 2, 3, \ldots$, and also to hyperbolic orbits of positive energy. This quadratic effect is of the non-secular type which could be calculated directly from the Kramers' dispersion formula, or from the perturbation technique for non-degenerate systems if all the amplitudes were known, and such a computation would, of course, give the same result as (3). In weak fields the relativity corrections may materially modify the intensity of individual Stark effect components, but not their aggregate intensity, either parallel or perpendicular to the field, as in virtue of this "spectroscopic stability" the quadratic term of non-secular type is invariant with respect to any perturbations which produce energy changes small compared to those involved in the (Lyman) transitions to states having higher values of the principal quantum number.

The mathematical detail of the rather technical proof that this invariance follows from the spectroscopic stability characteristic of the new quantum mechanics will be deferred until a later paper. We may simply mention that in summing the polarization given by the Kramers' dispersion formula over the various fine-structure levels (or in averaging ρ^2 over the various quantized orientations in the calculation of diamagnetism given below) we are led to an expression of the type form

$$\Sigma_k \Sigma_l q(kl)\, q(lk)$$

where $q(kl)$ is a matrix element, and where we sum k over all the component states of one multiple level and l over all the states of another. The essence of the proof consists in showing this sum invariant under a contact transformation of the type involved in reducing the secular part of the perturbative potential to a diagonal element. In the case of spacial degeneracy such a transformation corresponds to changing the axis of quantization. Born, Heisenberg and Jordan,[4] prove the invariance of the sum in case only the k or l level is multiple, but the extension to the case where they are both multiple occasions no difficulty. This invariance led the writer to predict in a recent note to *Nature*[6] that there would be no variation of the dielectric or diamagnetic susceptibility with pressure or field strength.

Thus despite the relativity corrections the dielectric constant has the same value 1.000229 in strong and weak fields. Similarly ϵ will be unchanged by other disturbances. We have not, for instance, mentioned the internal spins of the electron, but they respond to magnetic rather than electric fields. The "spin electron corrections" are thus mathematically coördinate with the numerically comparable relativity corrections, and without influence on the dielectric constant for the same reason. Furthermore, questions of spacial quantization do not enter, as the spectroscopic stability argument alluded to above shows that the dielectric con-

stant has the same value with random orientations as with the particular orientations allowed by the quantum conditions.

Dielectric Constant of Excited Hydrogen Atoms.—If the hydrogenic atoms are all in one particular excited state; i.e., all have some given value of $n \neq 1$, the part of the dielectric constant which is independent of the temperature may be determined from (2) by averaging over all the various allowed integral values of m_1, m_2, m_3 which are consistent with the given n. Each value of $m_3 \neq 0$ is to be counted twice because of the possibility of both left- and right-handed rotations about the axis of the field. Because of the relativity corrections which in weak fields introduce small frequencies in the denominator except when $n = 1$, the numerical magnitude of a_2 will be much greater than (2) when the field is very weak. It may, however, be proved that our average value of a_2 is invariant of the field strength, and can be calculated from (2) even in weak fields. The proof consists in noting that the non-secular part of $\bar{a}_2$ is invariant in virtue of the general spectroscopic stability argument alluded to above, while the secular part can be shown invariant by using the same method as that employed by Heisenberg and Jordan in proving the permanence of g-sums in the anomalous Zeeman effect (see especially the paragraph containing equation 17 on p. 269 of their article in *Zeits. f. Physik*, **37**, 269). In addition to the term proportional to $\bar{a}_2$, there will be another term in the dielectric constant arising from the fact that the Boltzmann temperature factor has different magnitudes for different component states. This second term is important only if the field is strong enough to produce a linear Stark effect. The complete dielectric constant is thus readily found to be

$$\epsilon - 1 = -4\pi N\bar{a}_2 + \frac{4\pi N\bar{a}_1^2}{kT} = \frac{Nh^6(15n^6 + 21n^4)}{128\pi^5 Z^4 e^6 \mu^3} + \frac{3Nh^4(n^4 - n^2)f(F)}{32\pi^3 Z^2 e^2 \mu^2 kT}$$

where $f(F)$ equals unity in strong fields, but vanishes in weak fields, in which always $a_1 = 0$.

Diamagnetic Susceptibility.—By a fundamental formula developed by Pauli,[7] the diamagnetic susceptibility per gram mol. is

$$\chi_M = -(e^2L/4\mu c^2)\Sigma\,\overline{\rho^2} \qquad (4)$$

where L is the Avogadro-Loschmidt number. We note by ρ, z, φ the cylindrical coördinates of an electron, with the magnetic field H as the z-direction. The summation in (4) extends over all the electrons in the atom or molecule. Equation (4) differs by a factor 2 from the original Langevin-Weber formula, as noted by Glaser[8] and Barnett,[9] and is much more satisfactory since Pauli assumes real electronic motions rather than hypothetical particles which can rotate only in one particular plane. The impression is usually conveyed that Larmor's theorem is basic to the derivation of (4), as it was so stated in Pauli's original paper. In reality the

assumption of Larmor's theorem is unnecessary, as (4) can be deduced from the very general dynamical principle that in a magnetic field the generalized momentum p_φ conjugate to φ must be defined[10] as $p_\varphi = \mu\rho^2\dot{\varphi} - (He\rho^2/2c)$ in order to preserve the Hamiltonian form of the equations of motion. The diamagnetic moment is simply a correction term resulting from the fact that the magnetic moment of an electron about the axis of the field is proportional to the ordinary angular momentum $\mu\rho^2\dot{\varphi}$ rather than to the generalized momentum p_φ given above which is canonically conjugate with respect to φ. For let M be the axial component of the part of the magnetic moment of an individual atom or molecule which arises from the orbital angular momentum of the electrons. Then $M = -{}^1/_2\,\Sigma e\rho^2\dot{\varphi}/c$ as $-e/2\mu c$ is the ordinary ratio of orbital magnetic moment to angular momentum. If we make the substitution $\mu\rho^2\dot{\varphi} = p_\varphi + (He\rho^2/2c)$ then

$$M = -(e/2\mu c)\Sigma p_\varphi - (e^2/4\mu c^2)\Sigma\rho^2 H.$$

The first term is simply the contribution of the orbital angular momentum of the electrons to the paramagnetic moment. The second term gives our equation (4) on multiplying by the Avogadro number and averaging over all atoms or molecules.

From this general mode of derivation it is seen that *Pauli's formula can be applied to molecules as well as atoms*, since on account of the factor $1/\mu$ we may neglect the contribution of the heavy nuclei to the diamagnetic moment. Furthermore *the advent of the spin electron does not destroy the applicability of* (4). There is, to be sure, a distinction between the ordinary definition of the angular momentum associated with the internal spins and the Hamiltonian definition in a magnetic field (a distinction similar to the orbital one described above, but with an internal dimension entering in place of ρ, etc.); nevertheless the resulting contribution to the diamagnetism is a mere nothing since the internal radius of the electron is extremely small. It is, of course, to be understood that the internal spin may give a large contribution to paramagnetism. It is, in fact, responsible for all the paramagnetism of normal hydrogen atoms, but we are concerned with the diamagnetic effects which always remain as residuals even in compounds (such as H_2) where the paramagnetic effects of individual electrons neutralize each other. To summarize, equation (4) applies far more generally than Larmor's theorem, but the literature is nevertheless honeycombed with misleading statements and formulas. We may note in particular a recent paper by Tartakowsky[11] which attempts to introduce the Landé g-factor into diamagnetism. This procedure appears erroneous to the writer because the general argument given above shows (4) holds even with anomalous precessions due to internal spins.

The method of deriving (4) shows that it will continue to hold in the new

quantum mechanics, as no non-commutative multiplications are involved. There is, however, the following marked superiority of the new dynamics. In the old quantum theory we could replace the average of $\rho^2 = x^2 + y^2$ over all atoms by two-thirds the time average of $r^2 = x^2 + y^2 + z^2$ for a single atom only if we assume a random spacial distribution of orbits, which makes the mean squares of x, y, z equal by symmetry; with spacial quantization the results would be different, as emphasized by Debye.[12] The spectroscopic stability characteristic of the new quantum mechanics (see a preceding paragraph in fine print) shows that *we may replace* $\overline{\rho^2}$ *by* $^2/_3\,\overline{r^2}$ *not only when there is random orientation, but also when there is spacial quantization* relative to the axis of the magnetic field, or even relative to any other axis of reference. This explains why recent experimenters find no "Glaser effect" (variation of susceptibility with pressure). It is indeed most comforting that in the new theory we can always write (4) in the form

$$\mathrm{X}_M = -(e^2L/6\mu c^2)\Sigma\,\overline{r^2} = -2.85\cdot 10^{10}\Sigma\,\overline{r^2} \qquad (5)$$

where it is clearly to be understood that the bar in (5) denotes a time average for an individual atom (or molecule) whereas in (4) the bar denotes an average over the different allowed orientations. We have tacitly supposed that all the atoms or molecules of the diamagnetic material have the same time average of r^2. This condition is clearly fulfilled by hydrogenic atoms in the normal state and also is satisfied by most other free atoms and approximately by molecules which are not unduly elastic. The excitation of states with different mean values of r^2 may, however, possibly be the explanation of the temperature variation of the diamagnetism of certain solid elements.

The time average of r^2 will be different for a hydrogenic atom in the new mechanics than in the old, and one of the main aims of the present paper is to communicate the new value. The requisite average value, which is, of course, a diagonal element of the matrix r^2, is readily calculated by methods due to Waller[1] (especially his Eqs. 35–37). We thus find

$$\overline{r^2} = a_0^2 n^2[^5/_2\, n^2 - {}^3/_2\, k(k+1) + {}^1/_2] \qquad (6)$$

where a_0 equals $h^2/4\pi^2\mu Ze^2$, the radius of a one-quantum orbit in the old theory, and where the azimuthal quantum number k assumes the values $0,1, \ldots, n-1$. The old quantum theory would give

$$\overline{r^2} = a_0^2 n^2[^5/_2\, n^2 - {}^3/_2\, k'^2]$$

where k' is one unit larger than the new k.

For hydrogen atoms in the normal state we have $Z = 1$, $n = 1$, $k = 0$ and equations (5, 6) then give $\mathrm{X}_M = -2.42 \times 10^{-6}$, a value three times that given by the old quantum theory. The diamagnetic susceptibility 2.93×10^{-6} is deduced for atomic hydrogen by Pascal[13] from experimental data

on organic compounds. He assumes that their susceptibilities are the sum of those of the constituent atoms plus a constant characteristic of the type of chemical bond. Pascal's value is over 250% higher than that given by the old theory (assuming the field too weak for spacial quantization), but only 22% higher than that given by the new. This must be regarded as a decisive confirmation of the new mechanics at the expense of the old. The 22% discrepancy is not surprising since atomic susceptibilities clearly cannot be calculated accurately by applying the method of mixtures to molecular data.

Application to Helium and Molecular Hydrogen.—Since direct measurements have apparently to date been prevented by the difficulty of obtaining pure monatomic hydrogen, perhaps the best experimental method of testing the theoretical susceptibilities is by the data for helium and molecular hydrogen. We shall assume that each helium atom or hydrogen molecule has the same susceptibility, energy, etc., as two hydrogenic atoms in the state $n = 1$ and with the effective nuclear charge Z. The theoretical energy, dielectric constant and diamagnetic susceptibility are then, respectively,[14] $W = 27.1 \cdot Z^2$ volts, $\epsilon = 1 + 0.000458Z^{-4}$, $\mathrm{X}_M = -4.84 \times 10^{-6}Z^{-2}$. The experimental values are: for He, $W = 78.8$, $\epsilon = 1.0000693$, $\mathrm{X}_M = -1.88 \times 10^{-6}$; for H_2, $W = 31.3$, $\epsilon = 1.000273$, $\mathrm{X}_M = -3.94 \times 10^{-6}$. On equating the theoretical and experimental values of W, ϵ and X_M, respectively, we have three independent estimates of the effective nuclear charge, and the good agreement between the three is a confirmation of the theory. The results are given in the following table:

EFFECTIVE NUCLEAR CHARGE Z

FROM	ENERGY	DIELECTRIC CONSTANT	DIAMAGNETISM
He	$Z = 1.71$ (1.71)	$Z = 1.603$ (1.10)	$Z = 1.607$ (0.93 [15])
H_2	$Z = 1.08$ (1.08)	$Z = 1.14$ (0.78)	$Z = 1.11$ (0.64 [15])

The values of Z which would have been obtained had the susceptibilities been calculated with the old quantum theory are included in parenthesis after the new values. *The results obtained with the new theory are obviously vastly more consistent and reasonable than those with the old, and this gratifying fact must be added to the many other successes of the Heisenberg-Schrödinger mechanics.* Exact agreement between the various estimates of Z is not to be expected, since introduction of a screening constant is clearly but a crude representation of the interplay between the two electrons. The closer agreement of Z calculated from the dielectric constant with that calculated from diamagnetism than with that calculated from energy is to be expected, as W, X_M and ϵ are all quadratic in the amplitudes but involve the frequencies to the powers $+2$, 0 and -1, respectively, and so any divergence of frequencies from hydrogenic values will cause more spread between values of Z calculated from ϵ and W than from ϵ and X_M. It is thus tempting to calculate diamagnetic susceptibilities with the

values of Z deduced from dielectric constants. The values 1.88×10^{-6} and 3.74×10^{-6} are then obtained for the diamagnetic susceptibilities of He and H_2, respectively, in remarkably close agreement with the experimental values 1.88×10^{-6} and 3.94×10^{-6} quoted above. The discrepancy may, however, be greater than apparent at first sight because the experimental values of X_M which we employ may be in error, especially in H_2. We have assumed Hector and Wills' values,[16] which are the only reliable data at present available for He, and which in the case of H_2 reassuringly agree within 2 per cent with an independent determination by Soné.[17] On the other hand tentative measurements by Lehrer[18] on hydrogen, which he states may be inaccurate, give $X_M = -5.1 \times 10^{-6}$ which requires $Z = 0.98$, while an even greater diamagnetic susceptibility for H_2 is reported in a preliminary communication by Hammar.[19] The low value of Z required by the newer measurements on hydrogen is possibly an argument against them.

We shall not attempt in the present paper to extend the numerical calculation of susceptibilities to atoms or molecules with more than two electrons. Such computations of dielectric constants would show a certain amount of resemblance to those of J. H. Jones[20] in the old quantum theory. The results would perhaps be less significant than those for H_2 and He because the difference between the new and old mechanics is less for large quantum numbers and also because complicated atoms approximate less closely to hydrogenic conditions.

* *Note Added in Proof.* Last month's issue of these PROCEEDINGS, **12**, 639, contains an experimental determination of the dielectric constant of atomic hydrogen by R. M. Langer, who finds $\epsilon - 1$ to have half the value for molecular hydrogen. This gives $\epsilon = 1.000136$ for atomic hydrogen, which does not agree with the theoretical value 1.000229 within his tentative estimate ± 0.000034 of the experimental error. The directness and unambiguity with which the quantum mechanics give $\epsilon = 1.000229$ suggest that the error in the very difficult experiment undertaken by Dr. Langer is larger than his estimate, possibly due to a smaller concentration of atomic hydrogen than ordinarily anticipated. Dr. Langer also includes an interesting second paper (*Ibid.*, **12**, 644) in which he endeavors to calculate the dielectric constant of atomic hydrogen with the new quantum mechanics by summing the various terms in the Kramers' dispersion formula. The theoretical value $\epsilon = 1.000338$ which he obtains appears to the writer clearly erroneous because it assumes that all of the dispersion comes from the absorption band beyond the head of this series. Hence the Kramers' dispersion formula should be integrated over the continuous spectrum as well as summed over the monochromatic part, but this would be a more laborious way of getting the same result as obtained above directly from the quadratic Stark effect.

[1] I. Waller, *Zeits. f. Physik*, **38**, 635, 1926.
[2] P. S. Epstein, *Physic. Rev.*, **28**, 695, 1926.
[3] Evelyn Aylesworth, *Physic. Rev.*, **28**, 847A, 1926.
[4] Born, Heisenberg and Jordan, *Zeits. f. Physik*, **35**, 589–590, 1926.
[5] E. Schrödinger, *Ann. d. Physik*, **80**, 446, 1926.
[6] J. H. Van Vleck, *Nature*, **118**, 226, 1926.
[7] W. Pauli, Jr., *Zeits. f. Physik*, **2**, 201, 1920.

[8] A. Glaser, *Dissertation*, Münich, 1924.

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[10] This can, for instance, be seen from Darwin's Lagrangian and Hamiltonian functions for atoms in magnetic fields, *Phil. Mag.*, **39**, 537, 1920.

[11] P. Tartakowsky, *Zeits. f. Physik*, **34**, 216, 1925.

[12] P. Debye, *Physic. Rev.*, **25**, 586A, 1925.

[13] A. Pascal, numerous references listed in *Jahr. d. Rad. and Elektr.*, **17**, 184, 1920.

[14] The value $\epsilon = 1.0000693$ for He is that obtained by Herzfeld and Wolf, *Ann. d. Physik*, **76**, 71 and 567, 1925, by extrapolating refractive indices to infinite wave-length, and is probably more accurate than direct determinations. The value $\epsilon = 1.000273$ for H_2 is that obtained directly by Tangl. This value apparently is for 0°C. rather than 20°C., contrary to the statement in the Landolt-Bornstein tables (5th ed., p. 1041) and so is in good agreement with dispersion data.

[15] If the field were strong enough for spacial quantization, the values of Z obtained from diamagnetism with the old quantum theory would be 1.14 and 0.78 instead of 0.93 and 0.64. We suppose the electric but not the magnetic field adequate for spacial quantization in the old theory.

[16] Wills and Hector, *Physic. Rev.*, **23**, 209, 1926; Hector, *Ibid.*, **24**, 418.

[17] Soné, *Phil. Mag.*, **39**, 305, 1920.

[18] E. Lehrer, *Ann. Physik*, **81**, 229, 1926.

[19] C. W. Hammar, *Proc. Nat. Acad. Sci.*, **12**, 594 and 597, 1926.

[20] J. H. Jones, *Proc. Roy. Soc.*, **105A**, 650, 1924.

ON THE EQUILIBRIUM BETWEEN RADIATION AND MATTER

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1. *Introduction.*—The theories of stellar evolution of Eddington, Jeans and Russell apparently necessitate the transformation of matter into radiation in order to account for the great life span of the sun and other stars. Such a process, however, immediately implies the existence of the reverse change of radiation into matter, and thus leads to the possibility of an equilibrium between these two forms of energy under suitable conditions of concentration and temperature.

By applying the laws of thermodynamics to an equilibrium mixture of radiation and matter Stern[1] has attacked this problem in a very stimulating and original manner, and derived an expression for the concentration of perfect gas which would be in equilibrium with radiation at any given temperature. He obtains the surprising result that, even at a temperature of one hundred million degrees, only one electron per cubic centimeter could be present at equilibrium. For a mixture containing equal numbers of electrons and protons, such as would presumably have to form from radiation in order to maintain electrical neutrality, the equilibrium concentrations would be even enormously lower. This result seems somewhat

unsatisfactory, since it might seem reasonable to expect that the change of matter into radiation in the interior of a star would take place at a rate high enough to maintain a concentration of matter not greatly in excess of equilibrium. Taking the sun as an example, we then find, however, that its mean density of 1.4 grams per cubic centimeter and internal temperature of the order of forty million degrees are very far from being in agreement with the Stern result.

The purpose of the present note is as follows. To obtain his derivation Stern has had to apply all three laws of thermodynamics, not only the dependable first and second laws of classical thermodynamics, but also the less certainly formulated third law or Nernst heat theorem. It will be shown below, however, that by applying merely the first two laws of thermodynamics it is possible to obtain an equation connecting concentration and temperature of the same form as Stern's, but containing, as might be expected, a constant which cannot be evaluated from the principles of classical thermodynamics. It will then be shown that the introduction of the third law in the form used by Stern leads indeed to his value of this constant. Arguments will be presented, however, which indicate that there might be a more correct manner of introducing the third law which would lead to a very much higher value of this constant.

2. *The Pressure-Temperature Relation for Matter and Radiation.*— Consider a cylinder provided with a sliding piston and with walls which can be kept at a constant temperature and can interchange radiation with the space inside. Let us start with zero volume behind the piston and with the walls at temperature T, and then withdraw the piston until the volume v is included behind it. If the process is carried out reversibly with respect to the emission of radiation and with respect to the equilibrium between radiation and matter, the work done will be

$$W = pv \tag{1}$$

where p is the combined pressure of matter and radiation at temperature T, and the heat absorbed will be

$$Q = E + pv \tag{2}$$

where E is the total energy of the matter and radiation thus produced inside the cylinder. Hence the combined entropy of this matter and radiation will be

$$S = \frac{E + pv}{T}. \tag{3}$$

We have now, however, as the condition of thermodynamic equilibrium that the entropy of our system must be a maximum for any variations at constant energy content and constant volume, and hence, taking T

and v as the independent variables, can write the simultaneous equations

$$\delta S = \left(-\frac{E + pv}{T^2} + \frac{1}{T}\frac{dE}{dT} + \frac{v}{T}\frac{dp}{dT}\right)\delta T = 0$$

$$\delta E = \frac{dE}{dT}\delta T = 0$$

and by combination obtain the result

$$v\frac{dp}{dT} = \frac{E + pv}{T}. \tag{4}$$

This is a general expression for the dependence of pressure on temperature in the case of any equilibrium mixture of matter and radiation. The equation can also be simply and directly derived from the Carnot principle, since a reversible cycle operating between temperatures $T + dT$ and T can be set up in which vdp is the work done and $E + pv$ the heat transferred from the upper to the lower temperature.

3. *The Pressure-Temperature Relation for Matter in Equilibrium with Radiation.*—If now we consider a temperature at which the concentration of matter is low enough so that the energy of the radiation is given by the Stefan-Boltzmann equation

$$E_R = avT^4 \tag{5}$$

and its partial pressure by the equation

$$p_R = {}^1/_3 aT^4 \tag{6}$$

we may evidently write for the radiation alone an equation of exactly the same form as (4), namely

$$v\frac{dp_R}{dT} = \frac{E_R + p_R v}{T}. \tag{7}$$

Hence by subtraction from equation (4) we can then obtain for the *matter alone* an equation of that same form, namely

$$v\frac{dp_M}{dT} = \frac{E_M + p_M v}{T} \tag{8}$$

connecting the partial pressure of this matter with its energy and temperature.

4. *Concentration of a Perfect Gas in Equilibrium with Radiation.*—In case we consider the matter to be present in the form of a perfect gas, we can easily integrate equation (8). We have for the pressure-volume product

$$pv = NkT \tag{9}$$

where N is the number of atoms present. And for the total energy including that necessary to form N atoms of mass m, can write

$$E = \frac{3}{2} NkT + Nmc^2. \tag{10}$$

Substituting in (8) we obtain

$$kT\frac{dN}{dT} + Nk = \frac{3}{2} Nk + \frac{Nmc^2}{T} + Nk \tag{11}$$

and integrating easily find our final expression

$$\frac{N}{v} = bT^{3/2}e^{-\frac{mc^2}{RT}} \tag{12}$$

where b arises from the constant of integration. This equation for the number of atoms present at equilibrium has the same form as that of Stern, but sets no requirements on the magnitude of the constant b.

5. *Discussion of the Constant b.*—To obtain a value of the constant b, we should have in some way to introduce the third law of thermodynamics, through a consideration of the absolute entropy of the gas. Returning, however, to the process discussed in section 2, it is evident, when we produce a mixture of radiation and matter by drawing out the piston through the volume v at temperature T, that the absolute entropy of the contents produced will be

$$S = \frac{E + pv}{T} \tag{13}$$

and this must be the *absolute entropy* of the radiation and matter formed, in the most precise sense that we can use that term, since at the start of the process the cylinder had no volume nor contents, and the above quantity is certainly the entropy that has flowed from the walls into the volume created behind the piston. Hence with concentrations of gas low enough so that the partial entropy of the radiation itself is given by an equation of the form of (13), we can write as an absolutely certain expression for the absolute entropy of the gas

$$S = \frac{E + pv}{T} = \frac{{}^3/_2 NkT + Nmc^2 + NkT}{T}. \tag{14}$$

Hence, if we had some suitable theoretical expression for the absolute entropy of our gas, we should at once be in a position to obtain an absolute value of the equilibrium concentration by equating with (14).

Now Stern's deduction is apparently equivalent to setting the above value for the absolute entropy of our gas equal to the Sackur-Tetrode expression for the entropy of a monatomic gas

$$S = kN \log \left\{ \frac{e^{5/2} v}{h^3 N} (2\pi mkT)^{3/2} \right\}. \tag{15}$$

Equating expressions (14) and (15) we then easily obtain the Stern result

$$\frac{N}{v} = \left(\frac{2\pi mk}{h^2} \right)^{3/2} T^{3/2} e^{-\frac{mc^2}{kT}}. \tag{16}$$

Such a use of the Sackur-Tetrode formula in this connection, however, seems far from justified since the theoretical derivations of this equation are based on considerations which do not contemplate the possibility of changes in the total number of atoms N. Thus if we follow Planck's[2] latest procedure, we may express the absolute entropy of a system by the equation

$$S = k \log P \tag{17}$$

where P is the total number of stationary states of the system whose energy does not exceed the total energy available; and, as he has shown, may then derive the Sackur-Tetrode expression for the entropy of a monatomic gas by considering all stationary states in which the *number of atoms* retains the *constant* value N and the *kinetic energy* does not exceed the value $E - Nmc^2$.

In the case of interest, however, the *number of atoms* could change at will from one to E/mc^2, the *kinetic energy* available for different configurations in the phase space changing at the same time from $E - mc^2$ to zero. Hence, it would seem that an application of the Planck method, correctly made for our purposes, would lead to a value of the absolute entropy much greater than that given by the Sackur-Tetrode formula, and hence to a value of the constant b in equation (12) much larger than that obtained by Stern.

Under the circumstances it may not be unreasonable to try to apply an equation of the form of (12) to the interpretation of astro-physical phenomena, leaving the absolute value of the constant as a problem for later investigation.

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[2] Planck, *Zs. Physik*, **35**, 155 (1925).

EXTENSION OF DOUBLET LAWS IN THE FIRST LONG PERIOD TO CHROMIUM AND MANGANESE[1]

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We have already shown[2] that the regular and irregular doublet laws may be applied to stripped atoms of the first five elements in the first long period of the periodic table. Recently secured data on chromium and manganese in the extreme ultra-violet have made it possible to extend these laws to Cr_{VI} and Mn_{VII}. In view of data reported by Lang[3] on elements in this period it seemed desirable to repeat our observations on vanadium to see whether the lines we had selected for the V_V doublet were correct or not, and to secure more accurate measurements of their wave-lengths.

A new photograph taken by using vanadium electrodes containing considerable carbon showed, unmistakably, only one line of the regular doublet at $\lambda = 1680.26$, the other being masked by several carbon lines, especially by a strong carbon doublet of the second order. A fairly strong vanadium line is found on this plate at $\lambda = 1722.51$ which is the one selected by Lang as the other member of this doublet. The separation between $\lambda = 1680.26$ and $\lambda = 1722.51$ does not yield a plausible screening constant, nor does it show the anticipated progression of $\Delta\lambda$. Since our earlier plate contained but few carbon lines it seemed evident that the line found at $\lambda = 1715.82$ with $\nu = 58281.1$, was due to vanadium. As that plate was not in excellent focus at this point a new photograph was made using a very pure specimen of vanadium which was very kindly given to us by Dr. A. S. King of the Mt. Wilson Observatory. This plate proved to be in excellent focus in this region with practically no carbon lines in evidence and showed a comparatively strong pair of lines in the same place as our original plate. With new secondary standards now available,[4] the same two lines were measured very accurately and are given at $\lambda\lambda = 1680.26$ and 1716.74 with the separation of $\Delta\nu = 1264.7$, this wave-length separation being only 0.02 A.U. different from that given in our first report.[2] The new measurements of the short wave-length line corresponds almost exactly with a line recorded earlier by Lang[5] in his measurements of the vanadium spark lines in vacuum. That our choice of lines for this doublet is correct is shown not only by the values of the screening constant and of $\Delta\lambda$ as can be seen in table 2, but also by the same frequency separation arising from jumps from the $4^2P_{1,2}$ levels to the 3^2D_2 level[6] whose frequencies we have already given in another report.[7]

As heretofore by an almost linear extrapolation of the frequency of these

doublets and by a similar extrapolation of the screening constant it was then possible to predict the approximate locations and the frequency separations of the corresponding doublets of Cr_{VI} and Mn_{VII}. Recent spectrograms of these elements, in sharp focus in these regions, enabled us to identify these doublets with a high degree of certainty. The frequencies of these lines are given in table 1.

TABLE 1

IRREGULAR N DOUBLETS

$4^2S_1 - 4^2P_1$

	LANG[3]	AUTHORS	DIFF.
K_I		12985.1	12206.5
Ca_{II}		25191.6	11373.7
Sc_{III}		36565.3	10968.0
Ti_{IV}	(47542)	47533.3	10716.6
V_V	(58048)	58249.9	10463.9
Cr_{VI}	(66756)	68713.8	10199.7
Mn_{VII}		78913.5	

TABLE 2

REGULAR N DOUBLETS

$4^2P_1 - 4^2P_2$

	$\Delta\nu$ LANG[3]	$\Delta\nu$ AUTHORS	$\sqrt{\Delta\nu/0.0456}$	s	$\Delta\lambda$
K_I		57.7	5.964	13.036	34.07
Ca_{II}		222.8	8.360	11.640	34.81
Sc_{III}	(474)	474.3	10.09	10.91	35.02
Ti_{IV}	(821)	817.5	11.57	10.43	(35.8)35.57
V_V	(1462)	1264.7	12.91	10.09	(42.3)36.48
Cr_{VI}	(2367)	1821.5	14.15	9.85	(51.3)37.58
Mn_{VII}		2464.7	15.25	9.75	38.38

Not only the consistent decrease in the values of the screening constant, and the small but systematic increase in $\Delta\lambda$, but also the uniform progression in the frequencies of the principal doublets $4^2S_1 - 4^2P_{1,2}$ throughout the first seven elements of this period give a beautiful confirmation of the validity of the regular and irregular doublet laws as applied to stripped atoms.

We wish to express our sincere thanks to Dr. I. S. Bowen for using the vacuum spectrograph at the Norman Bridge Laboratory to secure several of the spectrograms used in this report and to Mr. C. W. Gartlein for redetermining the frequency separation of the Ti_{IV} doublet whereby he finds the same separation as recorded before.[2]

[1] The assistance of a grant to the first author from the Heckscher Research Foundation of Cornell University which enabled us to obtain the results described in this report is gratefully acknowledged.

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IONIC MOBILITIES IN AMMONIA-HYDROGEN MIXTURES AND AN ANOMALOUS EFFECT OF AMMONIA

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Introduction.—In a previous paper[1] the writer has pointed out the advantages of studying mobilities in mixtures using hydrogen gas as one of the constituents. Some two years ago the writer and Miss Ashley[2] had determined mobilities in mixtures of ammonia and air. The results obtained were not completely in accord with the later work in mixtures inasmuch as they gave a variation of mobility with concentration different from that found in other mixtures studied. The results were, furthermore, not certain at low concentrations of NH_3 because of the relatively small difference in ionic mobilities for ions in air and NH_3, and it is just in the region of small concentrations of NH_3 that the results are interesting. Finally as brass electrodes were used and these were acted on by the NH_3 giving contact potential differences which had to be allowed for it was felt desirable to repeat measurements on an ammonia mixture free from these difficulties. Measurements were accordingly made in ammonia-hydrogen mixtures using gold-plated electrodes with the surprising results detailed below.

Experimental Procedure.—The ionization chamber, the gauze and the plate were those used for NH_3-air[2] mixtures except that the gauze and plate had been gold plated as they were in the H_2-ether mixture work.[1] The H_2 was purified as before. The ammonia was generated by dropping a concentrated NaOH solution onto Baker's analyzed C.P. NH_4Cl. It passed through a trap cooled to $-25°$C. by an alcohol bath, through a meter-long tube of finely powdered NaOH, and into a bulb cooled in liquid air where it was frozen out. The mixtures of the gases were obtained by exhausting the bulb of frozen NH_3 and the ionization chamber down

to pressures such that no discharge passed through a discharge tube with a large induction coil and 1.5 cm. spark gap in parallel. Then the pump was cut off and NH_3 distilled from the bulb to the requisite pressure by removing the liquid air. In some instances the first fraction distilled over was pumped off and only the central fraction used. This change in procedure made no apparent difference. The NH_3 was then shut off and hydrogen run in up to about 750 mm. pressure, all measurements being made between 730 and 760 mm., and corrected to 760 mm. The temperature of the chamber lay between 21°C. and 22°C. As in this mode of preparation the NH_3 was always accompanied by water, the exact degree of dryness of the gas might be questioned. The liquid NH_3 was always quite cold in distillation and probably very little water got over. Certainly in the lower percentages of NH_3 where the temperature was -70°C. this was the case.

The auxiliary field used was 7.5 volts per cm. The alternating potentials at which the ions crossed varied in this case from 17 volts per cm. to 60 volts per cm. The interpenetration[3] of the fields through the gauze caused notable changes in the mobilities for the two extreme alternating potentials. To avoid errors due to this in cases where 35 volts per cm. or more were used the auxiliary field was increased in proportion. All values given refer to the mobilities using an auxiliary field of 7.5 volts per cm. as against 20 volts per cm. alternating potential. With this gauze the generally accepted mobilities for air (1.4 cm./sec. for the positive 1.8 cm./sec. for the negative ions), are obtained when the auxiliary field is about 4 volts per cm. and the alternating potential about 30 volts per cm. Thus the fields used in this experiment would give mobilities in air of 1.6 cm./sec. for the positive ions and 2.05 cm./sec. This was proved by control measurements in air. The values of the mobilities in hydrogen and in NH_3 would thus be expected to be higher than the commonly assigned values in the above ratio. They are, however, slightly less than the true absolute values as determined by Loeb,[3,4] and Tyndall and Grindley.[5] Actually since mobilities in NH_3 and H_2 are measured under the same conditions, the absolute values assigned are of no interest to the problem. The present values can be reduced to the generally assigned values by multiplying by 0.88 and the correct absolute values of Loeb and Tyndall and Grindley can be obtained by multiplying by 1.05.

Anomalous Effect of NH_3 on Mobilities in H_2.—The mobilities measured in pure H_2 after carefully cleaning the ionization chamber with liquid soap and water, and rinsing repeatedly in distilled water were *6.07* cm./sec. and *9.36* cm./sec. for positive and negative ions, respectively. These correspond to 5.34 cm./sec. and 8.22 cm./sec. on the basis of 1.4 cm./sec. and 1.8 cm./sec. as the standard values in air. They agree fairly well with the values 5.94 and 8.90 found in the ether mixture experiments for

pure H_2 in a less carefully cleaned chamber. After introducing 3% of NH_3 and making measurements for three hours, the chamber was exhausted to the lowest pressure obtainable and filled with pure H_2 gas. The mobility was measured within an hour on both ions and the values *7.86* cm./sec. and *9.21* cm./sec. were obtained for positive and negative ions, respectively. This indicates that *while the negative ion mobility remained the same* (within the limits of error), *the positive ion had its mobility increased by the presence of a trace of* NH_3 in the chamber. Had this been due to a contact potential effect, the negative mobility would have been lowered and the positive mobility would have been raised by about the same amount. Subsequent control measurements with pure H_2 after NH_3 had been used (even after the vessel had been pumped for hours, filled with pure H_2 and pumped out again before measurement), showed essentially the same values of *7.88* cm./sec. and 9.21 cm./sec. Dry air was then admitted and mobilities in air were measured. The values *1.80* cm./sec. and *2.05* cm./sec. were observed showing that the abnormally high positive mobility (normal value 1.6 cm./sec. under these conditions), persisted in air. To remove the traces of NH_3 without opening up the chamber and changing conditions, air containing the fumes of HCl from an aqueous solution of HCl was passed over $CaCl_2$ and into the exhausted chamber. After 10 minutes it was removed. An hour's pumping and filling with dry air gave mobilities of *1.12* cm./sec. and *1.72* cm./sec. for the ions. Both these values are low as is to be expected in the presence of traces of HCl. The positive mobility, however, has its normal value relative to the negative mobility. Subsequent pumping and filling gave the values of *6.71* cm./sec. and *8.86* cm./sec. for pure H_2. It is seen that the trace of HCl removed the NH_3 and at once the positive mobility took on the low value relative to the negative mobility which usually is found in the absence of NH_3. The value 6.71 for positive ions is not quite as low as the value in the newly cleaned chamber. This is possibly attributable to the slow diffusion of traces of NH_3 back into the chamber through a defective stopcock in the hours of pumping. On re-admitting NH_3 the values *7.97* cm./sec. and *7.86* cm./sec. were later found for the positive ions in pure H_2, and 9.06 and 8.98 were found for the negative ions. It might be added that once when NH_3 was used to clean out HCl in the HCl-air mixture[6] experiments the anomalous effect of NH_3 on the positive ion had been noticed but seemed too fantastic in the presence of supposed contact effects to be further studied. In the earlier experiments in NH_3-air mixtures this effect was badly complicated by a *real contact potential* effect due to changes of the electrode surfaces. It was thus included in the contact potential correction and was not recognized. The foregoing results indicate without question that the *presence of exceedingly small traces of* NH_3 *in* H_2 *and air act to INCREASE the mobility of the positive ions by as*

much as 30% without affecting the negative ion. As will be seen subsequent addition of NH_3 gas (from 0.1 of 1 per cent on) lowers both mobilities by about the same amount. This is the first time that the effect of the *increase* of a mobility through the addition of a trace of gas has been observed.

This interesting result is capable of interpretation in the following fashion. Ammonia is a molecule that tends to add on a positive hydrogen nucleus to form a stable NH_4^+ ion. It is perfectly conceivable that the NH_3 molecule very readily attaches to the ionized positive hydrogen atom or molecule to form a stable ion analogous to the NH_4^+ ion in solution. In this form the ion is very stable and any further clustering about the NH_3 molecule plus its positively charged H atom or molecule, is impossible. The work of Erikson[7] shows that in N_2 the initial positive N_2 ion has the same mobility as the negative ion. After some time (0.02 second) its mobility drops to the accepted value. This he interprets in terms of the addition of some single molecule to form a bi-molecular cluster ion. If this molecule added were one of large size in the relatively impure gases usually worked with, the decrease in mobility would be appreciable. Substitution of such a molecule by a small NH_3 molecule where traces of the latter are present would then explain the whole effect. The effect of a bulky molecule such as one with a long carbon chain, compared to a smaller molecule has been shown already in the work of Tyndall[8] and Phillips for the case of the aliphatic group of alcohols in alcohol-air mixtures. Whether the recent peculiar results of Erikson[9] on acetylene ions in air can be explained on a selective condensation of C_2H_2 on the positive ion to the exclusion of other molecules present in the air used cannot as yet be answered. It is proposed to test this point shortly.

With the hope of finding whether the NH_3 molecule would "protect" the positive ion from the lowering effects of ether the mobility of the ions in a mixture of hydrogen with 0.53% of ether and 0.53% of NH_3 was measured. In this mixture mobilities of 3.61 cm./sec. and 5.81 cm./sec. were observed. These may be a little low due to the effect of interpenetration of the fields which were not corrected for. The corrected values would be about 4.0 cm./sec. and 6.1 cm./sec. The correction is necessary for the alternating fields were 35 and 55 volts which call for corrections of this magnitude. If the fractional lowering of the mobilities of ions in H_2 in the presence of a *trace* of NH_3 vapor be calculated from the observed curves for NH_3-H_2 and ether-H_2 mixtures the mobilities should have been 5.37 cm./sec. and 6.26 cm./sec. If one calculated the lowering, assuming that the positive ion in this mixture had the initial value of positive ions in pure H_2, with no NH_3 present, the values would have been 4.08 cm./sec. and 6.26 cm./sec. These values, accepting the correction

for interpenetrating fields given, would be approximately equal to the observed values. The conclusion, therefore, is that *the ether molecule attaches to the positive ion in preference to the NH_3 molecule.* The mobility in the mixture then is due to the superposed retarding effect of NH_3 and ether molecules in H_2 gas on a positive ion carrying an ether molecule (or molecules).

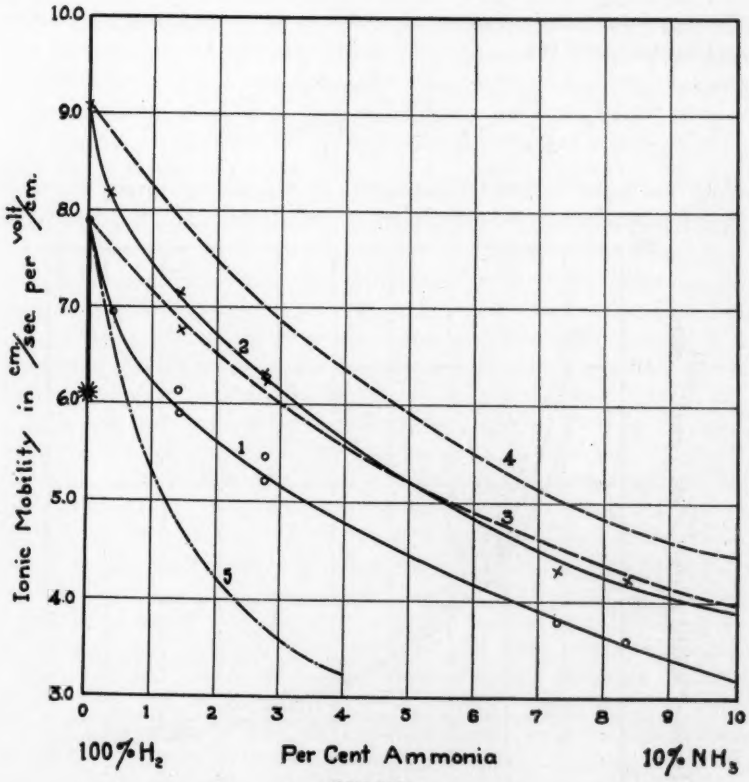

FIGURE 1

Mobility in NH_3-H_2 mixtures for values of 0% NH_3 to 10% NH_3

These results *unquestionably point to the permanent attachment of at least one* (perhaps more), *foreign molecules to the positive ions in H_2 and air,* thus bearing out Erikson's[10] contentions.

Mobilities in Mixture.—In figure 1, curves 1 and 2, the mobilities of positive and negative ions, respectively, in NH_3-H_2 mixtures are given as a function of the percentage composition for percentages of NH_3 from 0 to 10%. The mobility of the positive ion in *uncontaminated* H_2 is shown

by the star. The points are those actually observed. The dotted curves 3 and 4 are the theoretically [11,1] computed positive and negative mobilities given by the equation,

$$K_C = \frac{K_{NH_3}K_H}{\frac{(100 - C)K_{NH_3} + CK_H}{100}}.$$

The single dot and dash curve, curve 5, is a curve computed for positive ions by the equation,

$$K_C^1 = \frac{K_{NH_3}K_H}{\sqrt{\frac{(100 - C)K_{NH_3}^2 + CK_H^2}{100}}}$$

which was found to hold for the results in NH_3-air[2] mixtures. Figure 2 gives the same curves for percentages of NH_3 from 10 to 100%, the numbers of the curves being the same as those in figure 1. Here curves 2 and 3 coincide so closely that they are represented by one line.

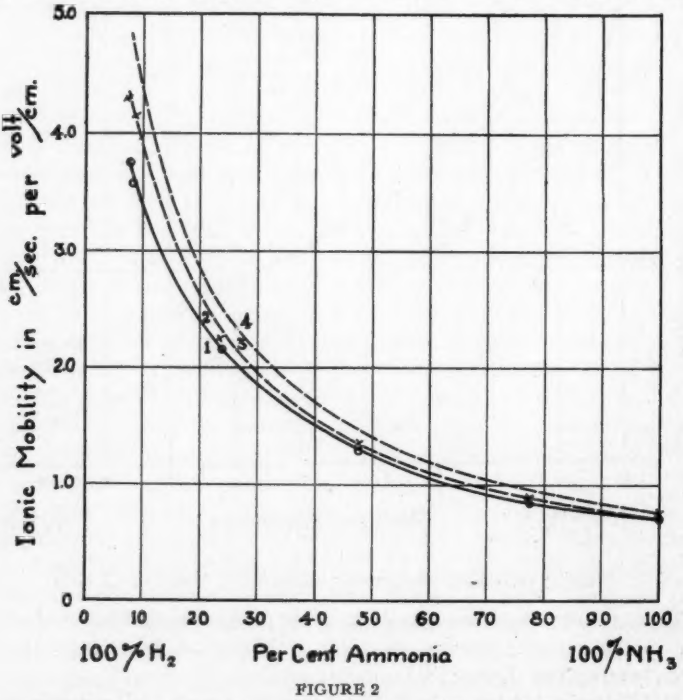

FIGURE 2

Mobility in NH_3-H_2 mixtures for values of 10% NH_3-100% NH_3.

It is seen that the mobilities of the ions in H_2, K_{H^+} and K_{H^-} are taken as 7.93 and 9.09 cm./sec., respectively, corresponding to mobilities 1.6 cm./sec. and 2.05 cm./sec. for the ions in air. In NH_3 the values of the mobilities $K_{NH_3^+}$ and $K_{NH_3^-}$ are taken as 0.707 cm./sec. and 0.77 cm./sec. on the same air standard. Reduced to the usual air standard they would be 0.62 cm./sec. and 0.67 cm./sec. These differ somewhat from the values found by Loeb and Ashley[2] in the NH_3-air work, where the values 0.50 and 0.65 were obtained. The difference comes from the correction for contact potential which, as it included the anomalous effect in NH_3 described above, *gave an over-correction*. The true values on the accepted scale are, therefore, 0.62 cm./sec. and 0.67 cm./sec. Wahlin's[12] value of 0.78 cm./sec. obtained for negative ions in the absence of a gauze is probably the true absolute value and is in agreement with the value 0.77 found in this work for the high auxiliary field.

It is seen at once that the observed points fall below the theoretically computed curves. In this case for the negative ion as well as the positive ion they deviate more from the theoretical curves at the lower concentrations of NH_3. In the case of NH_3, however, the observed points do not fall on the theoretical curves at the higher concentrations as is the case in ether. The tendency for the observed curves throughout their course to follow another law than the theoretical law is here definitely indicated.

As is seen from curve 5 the second equation found to hold for NH_3-air[2] mixtures falls way below the observed values for positive ions in NH_3-air mixtures. The law found for NH_3-air mixtures is, therefore, not applicable here, although the observed curves deviate from the theoretical curves in the same sense as observed for NH_3-air mixtures. This is an important point brought out by the use of H_2 instead of air. In the NH_3-air mixture work the comparatively small difference between NH_3 mobilities and air mobilities, together with the over correction for contact potential, made an accurate test of the mobility variation with composition impossible.

Summary and Conclusions.—The results obtained definitely indicate that *traces* of NH_3 in H_2 and air make a protective cluster of one or more molecules about the positive ion that keeps other bulky molecules from attaching and raises the mobility of the positive ion to nearly that of the negative ion. The NH_3 molecule does not protect the positive ion against ether molecules present in any numbers. In that case the ether molecule produces the clustering effect and the NH_3 molecule acts only to slow up the ion due to its dielectric action and change of concentration.[6]

Both positive and negative ions for percentages of NH_3 more than 0.3% have their mobilities reduced by further addition of NH_3 and more rapidly than the common law of mixtures would lead one to expect. The law of decrease is, however, not the same as that found for NH_3-air mixtures, whose accuracy may be doubtful in view of the possibility of over-correc-

tion for contact effects and of the small difference between NH_3 and air mobilities. It points, however, in the same direction as the NH_3-air work; namely, to a preferential change in concentration of NH_3 molecules in the neighborhood of both ions. This effect is noticeable even at high concentrations of NH_3.

One thus sees in the negative ion the effect of the dielectric constant of NH_3 on the mobility without clustering while in the positive ion one has both effects superimposed, the clustering in this case increasing the mobility by its protective action, the dielectric constant causing a lowering as is to be expected. The results again indicate the importance of *minute* traces of gases in ion mobility work and thus indicate the necessity of extreme care in controlling the purity of the gases.

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NOTE ON THE CORRESPONDENCE PRINCIPLE IN THE NEW QUANTUM THEORY

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1. *Synopsis.*—In this note it is shown that if the coördinate matrix of the new quantum theory is expanded in powers of Planck's constant, h:

$$X(mn) = X_0(mn) + hX_1(mn) + h^2 \ldots\ldots \qquad (1)$$

then $X_0(mn)$ approaches the coefficient of the $(m-n)$th harmonic in the Fourier expansion of the motion in the old theory when n approaches infinity. For large values of the quantum numbers, the Fourier coefficients thus constitute a first approximation to the matrix terms, which is in accordance with the experimental evidence embodied in Bohr's correspondence principle.

2. *The Expansion of the Characteristic Functions in Powers of h.*—The fundamental equation of the new quantum theory, for the case of one degree of freedom, is

$$-\frac{k^2}{2m}\frac{d^2\psi}{dx^2} + [U(x) - E]\psi = 0, \quad k = \frac{h}{2\pi} \tag{2}$$

where U is the potential energy function of the older theory and E is the energy of the system. The discrete values of E are determined by the condition that ψ be single valued and have no singularities for real values of x. This condition can only be satisfied for certain values of E.[1]

If we set $\psi = e^{\phi/k}$ (2) becomes

$$-\frac{1}{2m}\left(\frac{d\phi}{dx}\right)^2 + U - E - \frac{k}{2m}\frac{d^2\phi}{dx^2} = 0.$$

Expanding ϕ in powers of k: $\phi = \phi_0 + k\phi_1 + \ldots$, we obtain the following equations to determine the successive members of the series:

$$-\frac{1}{2m}\left(\frac{d\phi_0}{dx}\right)^2 + U - E = 0$$

$$2\frac{d\phi_0}{dx}\frac{d\phi_1}{dx} + \frac{d^2\phi_0}{dx^2} = 0.$$

The first of these can readily be seen to have as its solution $\phi_0 = iS(xJ)$ where S is the Hamilton-Jacobi function and J the action variable of the older theory.[2] In every case thus far studied, the condition on ψ is satisfied only when[3]

$$J = \left(n + \frac{1}{2}\right)h = J_n \qquad n = 0, 1, 2\ldots.$$

Hence the sequence of characteristic functions is

$$\psi(x,n) = e^{\frac{2\pi i}{h}\left[S(xJ_n) + \frac{h}{2\pi}\cdots\right]} \tag{3}$$

where $\frac{h}{2\pi}\ldots$ represents the aggregate of terms of order of magnitude h or smaller.

3. *The Relation of the Matrix Terms to the Fourier Coefficients of the Older Theory.*—The matrix corresponding to the coördinate x may be defined by the equation[1]

$$x\psi(nx) = \sum_{m=0}^{\infty} X(mn)\psi(mx).$$

Substituting the expression (3)

$$x = \sum_{m=0}^{\infty} X(nm)\, e^{\frac{2\pi i}{h}\left[S(xJ_m) - S(xJ_n) + \frac{h}{2\pi}\cdots\cdots\right]}. \qquad (4)$$

If the exponent of each term in (4) be expanded in powers of $J_m - J_n = (m - n)h$, and if $X(mn)$ is also expanded in powers of h, the result is

$$x = \sum_{m=0}^{\infty}{}' X_0(mn)\, e^{2\pi i(m-n)\frac{\partial S}{\partial J}} + R(h). \qquad (5)$$

where

$$\lim_{h=0} R(h) = 0.$$

On the other hand, in the older theory, the expression of x as a function of time is obtained as

$$x = \sum_{p=-\infty}^{+\infty} A(p)\, e^{2\pi i p\omega t}$$

where $\omega = \dfrac{\partial E}{\partial J}$. This equation is identical with the equation[3]

$$t = \frac{\partial S}{\partial E}.$$

Hence the equation

$$x = \sum_{-\infty}^{+\infty} A(p)\, e^{2\pi i p \frac{\partial E}{\partial J}\frac{\partial S}{\partial E}}$$

is an identity in x. If $p = (m - n)$, it may be written

$$x = \sum_{m=0}^{\infty} A(m-n)\, e^{2\pi i(m-n)\frac{\partial S}{\partial J}} + T(n) \qquad (6)$$

where

$$\lim_{n=\infty} T(n) = 0.$$

Comparing (5) and (6), we see that

$$\lim_{h=0,\ n=\infty} X(mn) = A(m - n)$$

and we obtain the theorem:

As h approaches zero and n approaches infinity in such a way that $\left(n + \frac{1}{2}\right)h = J$, the matrix term $X(mn)$ approaches the coefficient of the $(m - n)$th harmonic in the Fourier expansion of the classical motion.

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[3] Cf. Wentzel, *Zs. Physik*, **38**, 518 (1926).

MINIATURE-α—A SECOND FREQUENTLY MUTATING CHARACTER IN DROSOPHILA VIRILIS

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The first frequently mutating character observed in *Drosophila virilis* was the body character "reddish,"[1] which was found to mutate frequently to wild type. The time of occurrence of the mutations was found to be limited to the maturation divisions of heterozygous females; no mutation in somatic cells was ever detected. Also no mutations were observed in male or homozygous reddish females. The frequency with which reddish mutates to wild type varies greatly in the different individuals obtained from the same parents, which makes it possible to select for low or high frequency of mutations. In that way, from a mutating reddish, a constant reddish can be isolated very easily, or if the selection is carried in the other direction, a high frequency mutating line can be established.

Miniature-α wing character is the second frequently mutating character found in *Drosophila virilis*. The behavior of miniature-α differs in several respects from the behavior of reddish, but like reddish, miniature-α mutates to wild type.

Origin of Miniature-α.—A single miniature male was found among 177 flies of an experiment with reddish, and from this male all miniature-α flies were derived. In the F_2 generation from a mating between that male and several wild-type females, in addition to 258 wild-type females and 282 wild-type males, 11 miniature males were obtained. Those results showed that the character is inherited and indicated that it is sex linked.

Tests for Mutability.—In the first and the following F_2 generations and backcrosses with miniature-α very few miniature flies were obtained. One of the several explanations to account for this deficiency was the possibility that miniature-α reverts frequently to wild type. To test that possibility miniature was crossed with stocks carrying several sex-linked characters, since F_2 and backcross data showed that miniature-α is located in the sex chromosome. Miniature-α was also crossed with

miniature-1 (known since 1922 in *Drosophila virilis*) because of the close resemblance between these two characters.

In the F_1 generation of one of the crosses between miniature-1 female and miniature-α male 13 wild-type females and 17 miniature males were obtained, and in another cross 19 wild type and 16 miniature females and 28 miniature males were obtained. If miniature-α and miniature-1 were allelomorphs, all F_1 females would be expected to be miniature, and if these characters were not allelomorphs, all F_1 females would be expected to be wild type. In neither case would both wild-type and miniature females be expected in the F_1 generation.

Six miniature females obtained in the F_1 generation from the cross between miniature-1 female and miniature-α male were backcrossed to miniature-1 males. They gave a total of 181 miniature and 209 wild-type females, and 170 miniature and 174 wild-type males. In the case of sex-linked recessive characters in *Drosophila*, the male does not have any influence on the type of its male progeny. If the female is homozygous for a sex-linked character, all her male offspring are expected to show the character (except for rare exceptions due to non-disjunction), regardless of the type of male to which she is mated. Unexpected wild-type females obtained in the experiment might be attributed to contaminations. That is, however, very improbable because of the large numbers involved and the occurrence of wild-type females in six independent matings. The occurrence of wild-type males from miniature females, in such large numbers as obtained, is highly exceptional and cannot be accounted for in any usual way. Similar results were obtained when the same experiment was repeated several times.

Crosses were made involving miniature-α and sepia, yellow, scute, vermilion, forked, white, rough, small bristles-2, which are located throughout the whole map length of the sex chromosome. All of the characters, except miniature-α, behaved as expected, showing that there was no disturbance in the behavior of the sex chromosome. In these experiments a great deficiency or an entire absence of miniature-α flies was observed.

To determine which of the two indistinguishable miniature is responsible for the occurrence of wild-type flies in the crosses between miniature-1 and miniature-α matings were made by using miniature-1 forked-2 females. Since forked-2 is located close to miniature (there is about 8 per cent of crossing-over between them) it was expected that a great majority of forked flies would have miniature-1, and the majority of flies without forked would be miniature-α. The results of the crosses are presented in table 1, from which it can be seen that again as in the previous crosses between the miniatures, the F_1 generation included a considerable number of wild-type females. In the backcrosses between the F_1 miniature females and miniature-1 males, most of the forked flies were miniature

and the great majority of non-miniature flies were not forked. This indicates the normal behavior of miniature-1 (which came into the cross together with forked) and identifies the origin of the wild-type flies with miniature-α. The miniature classes, which are largely composed of miniature-α flies, are deficient when compared with miniature, forked classes which are largely miniature-1 flies. That deficiency is compensated by wild-type classes, supporting the assumption that wild-type flies originated by a change of a gene for miniature-α to a gene for wild type.

TABLE 1

PROGENY FROM CROSSES BETWEEN MINIATURE-1, FORKED-2 FEMALES AND MINIATURE-α MALES

PEDIGREE NUMBER	F_1 FEMALES			BACKCROSSES: F_1 *mi* FEMALES × *mi*/f_2 MALES							
				FEMALES				MALES			
	mi	+	TOTAL	*mi* f_2	f_2	*mi*	+	*mi* f_2	f_2	*mi*	+
4218	29	85	114	53	1	65	4	58	.	57	3
4220	67	3	70	116	10	42	102	104	7	34	89
Total	96	88	184	169	11	107	106	162	7	91	92

A miniature-α female mated with miniature-1 male gave 78 miniature females and 71 miniature and 5 wild-type males. Two miniature-α females mated to the males of the same type gave 78 miniature-α and 115 wild-type females, and 102 miniature-α and 55 wild-type males. Those experiments were repeated many times, giving similar results. Not only miniature females heterozygous for miniature-α, but also those homozygous for this character, produced wild-type males. Since the experiments involving several sex-linked characters, and especially those with miniature-1 forked, eliminated the possibility of chromosomal abnormalities being responsible for the occurrence of the unexpected wild-type flies, the most probable hypothesis which accounts for that abnormal behavior is the assumption that the gene for miniature-α mutates frequently to the gene for wild type.

Tests of Wild-Type Flies.—Wild-type flies obtained from miniature-α behave like any other wild-type flies. No mutation to miniature has yet been obtained in the progeny of those flies. Six wild-type males obtained from miniature-α females were mated to wild-type females. Among 10,422 F_2 flies no miniature-α flies were observed. Nine wild-type males were mated to miniature-1 females and all of 304 females obtained from these matings were wild type.

The Time of Occurrence of Mutations.—As already stated wild-type flies were obtained from miniature-α as follows:

(1) In F_1 females from crosses between miniature-1 females and miniature-α males.

(2) In F_2 progenies from miniature females of the same cross.

(3) In the progenies of homozygous miniature-α females.

This shows that miniature-α mutates to wild type at the formation of germ cells in miniature-α males and heterozygous and homozygous females.

Miniature-α mutates to wild type in somatic cells as well. Mosaic flies having part of the wing miniature and the other part wild type are very frequent and are easily distinguished. If miniature-α wings are examined under high magnification almost all of them show small areas of wild-type tissue.

The Frequency of Mutations.—The frequency of mutations of miniature-α to wild type can be changed by selection. Selection made for low frequency gave a practically constant miniature line, and selection for high frequency of mutation resulted in a line in which over 70 per cent of flies from miniature-α parents were wild type.

Discussion.—In its general behavior miniature-α can be compared with well-known, frequently mutating genes in plants which produce chlorophyll and anthocyan variegations. On the other hand, it differs strikingly from the gene for "reddish" (mentioned above) because it mutates in all stages of development. (The mutating period of reddish is strictly limited to the maturation division of heterozygous females.)

[1] Demerec, M., these PROCEEDINGS, **12**, 1926 (11–16).

GENETIC EVIDENCE OF A SELECTIVE SEGREGATION OF CHROMOSOMES IN SCIARA (DIPTERA)

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A cytological study of chromosome behavior during spermatogeneses in *Sciara* gave evidence indicating that a selective, instead of random, segregation of chromosomes takes place here, such that the maternal chromosomes separate bodily from the paternal ones.[1] The evidence also indicated that one parental haploid group was regularly cast off and not transmitted by the sperm.[2] From the behavior of two large "sex-limited" chromosomes, found only in males, it was postulated that the maternal chromosome group was the one regularly cast off.

A genetic test has now been made which reveals the actual behavior of one pair of chromosomes, and shows that, at least so far as this pair is concerned, the earlier inferences were correct in principle. The evidence comes from breeding experiments with a recessive mutant wing character "truncate" in *Sciara coprophila*, Lint., the inheritance of which is as follows:

A truncate female by wild-type male gives in F_1 all wild-type offspring, as expected. The heterozygous F_1 males, however, breed as if they were pure truncate—i.e., they transmit only the gene derived from the mother. From the reciprocal cross the F_1 males, although of the same genetic constitution, breed as if they were pure wild type—again transmitting only the gene derived from the mother. Similarly in subsequent generations a heterozygous male breeds as if he were homozygous for the maternal gene. This is equally true whether the mother is homozygous or heterozygous; i.e., it is independent of the somatic constitution of the mother.

The genetic behavior of the female is apparently of the "ordinary" type. A heterozygous female transmits both truncate and its wild-type allelomorph. Apparently segregation here is more or less at random, although wide deviations from a 1:1 ratio are sometimes observed.

The inheritance of truncate has been followed through four generations, including back crosses of F_1 flies of both sexes from reciprocal matings, and numerous back crosses of F_2 flies. The evidence, therefore, seems to leave no question that normally the chromosome pair considered here undergoes a selective type of segregation in the male—the maternal member always going to the pole which is to remain in the spermatocyte and be transmitted by the sperm. In the latter respect the actual behavior is the reverse of that postulated from cytological observations, although the principle involved is the same.

The data from these experiments also show that true fertilization takes place in *Sciara* in the production of females as well as of males—a feature which seemed doubtful from cytological evidence alone, and which necessitates the assumption of additional peculiarities in chromosome behavior (considered elsewhere).

Without attempting to present the evidence upon which the above conclusions are based, one experiment may be cited as an illustration of the type of behavior found here. This experiment began with reciprocal matings between wild type and truncate; (*a*) wild-type female by truncate male, and (*b*) truncate female by wild-type male. In both cases the F_1 flies were all wild type (73 and 126, respectively). Subsequent matings were then made as follows:

1. F_1 males from mating (*a*) were backcrossed to truncate females. These gave only wild-type offspring (90 flies).

2. F_1 flies from the reciprocal mating (*b*) were bred *inter se*. These gave an F_2 consisting of 59 wild type : 54 truncate.

3. Males (all brothers) from this F_2 were mated singly to females (all sisters) from mating 1. Eight of the males were wild-type and twelve truncate. The former gave only wild-type offspring (409 flies), while the latter gave in every case both wild type and truncate (total, 228 : 244).

The results of this experiment (selected because it illustrates several

different types of matings) deviate widely from those expected on the basis of ordinary Mendelian inheritance, but are readily interpreted when allowance is made for the aberrant chromosome behavior in the males.

[1] Metz, C. W. 1925. "Chromosome Behavior in Sciara (Diptera)," *Anat. Rec.*, **31**, 346. Cf. also *Science*, **61**, 212 and **63**, 190.

[2] A detailed account is in press in the *Zs. ind. Abst. Vererb.*

DECOMPOSITION OF AMMONIA BY OPTICALLY EXCITED MERCURY ATOMS

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Introduction.—Recently there have been studied various chemical reactions sensitized by mercury vapor activated by the absorption of the line 2537 Å. These reactions are often so complicated as to make their mechanism difficult to elucidate. For instance, in the case of the hydrogen-oxygen reaction, which has been studied by ourselves and later by others, there is the possibility of activating oxygen as well as hydrogen and the possibility of forming hydrogen peroxide and ozone as well as water; this reaction is further complicated by the removal of mercury vapor through oxidation. Consequently, it was desirable to study a reaction which *a priori* seemed relatively simple. It was known that ammonia can be decomposed photochemically[1] by the absorption of light of wave-lengths 2025 to 2140 Å. Preliminary experiments showed that ammonia decomposition could also be brought about by the wave-length 2537 Å when mercury vapor was present, although ammonia itself does not absorb this line. The decomposition products of both the sensitized and unsensitized reactions have been examined.

Experimental Procedure.—Light from a quartz-mercury arc was allowed to shine on two concentric cylindrical quartz tubes. The inner tube was connected to a mercury-vapor pump and McLeod gauge and could be filled with ammonia at any desired pressure. In the annular space between the two tubes a solution could be placed to serve as a light filter. In a side-arm connected to the inner quartz tube was placed about 2 cc. of mercury, in order to give a partial pressure of mercury corresponding to its vapor-pressure at room temperature. Through a liquid-air trap a quartz-fibre manometer, whose purpose was the examination of the reaction-products, was also connected to the inner tube. This manometer had two quartz fibres fused together at one end to prevent Lissajou

motion, as suggested by Coolidge.[2] At the low gas pressures used here the logarithmic decay constant λ of the amplitude of vibration of the fibre has been shown by Haber and Kerschbaum[3] to be given by the relation

$$A.\sum(p\sqrt{M}) + B = \lambda = \frac{0.693}{\tau_{1/2}}.$$

Here A and B are apparatus constants, $\tau_{1/2}$ is the time required for the vibration to decrease to half amplitude, and $\sum(p\sqrt{M})$ is the sum of the products of the partial pressures of the separate components of the gas into the square-roots of their molecular weights. The gauge was calibrated against the McLeod gauge using both hydrogen and air, care being taken to exclude mercury vapor; and a nearly straight-line calibration curve was drawn having λ as ordinates and $\sum(p\sqrt{M})$ as abscissae.

SENSITIZED DECOMPOSITION OF AMMONIA

INITIAL PRESSURE OF NH_3	FINAL PRESSURE OF RESIDUAL GAS	$\Sigma(p\sqrt{M})$	MOL.-FRACTION OF HYDROGEN
0.138 mm.	0.0055 mm.*	0.0124 mm.	0.786
0.137	0.0069	0.0171	0.727
0.770	0.0171*	0.0388	0.782
0.137	0.0097	0.0254	0.691
0.238	0.0165	0.0431	0.694
0.490	0.0286	0.0761	0.681
0.191	0.0124	0.0328	0.685
0.488	0.0348	0.0937	0.673
0.429	0.0303	0.0830	0.660
0.461	0.0190	0.0551	0.619
0.450	0.0234	0.0613	0.690
		Mean	0.699

* Time of illumination, 5 minutes.

DIRECT PHOTOCHEMICAL DECOMPOSITION OF AMMONIA

0.140	0.0062	0.0109	0.913
0.132	0.0075	0.0127	0.930
0.229	0.0134	0.0238	0.908
0.497	0.0232	0.0395	0.925
0.480	0.0327	0.0700	0.814
0.426	0.0226	0.0458	0.843
0.497	0.0289	0.0601	0.835
14.0	0.147	....	....
0.135	0.0058	0.0132	0.789
		Mean	0.870

The experimental procedure was as follows. After pumping out the whole system, ammonia manufactured by the Fixed Nitrogen Research Laboratory was admitted through suitable stopcocks and condensed in the liquid-air trap, the non-condensible impurities being then pumped off.

With the apparatus closed from the pumps, the liquid air was removed, and the ammonia pressure was measured on the McLeod gauge. In case the sensitized reaction was to be studied, a 1.7 per cent solution of tartaric acid was placed in the 5 mm. annular space to filter off radiation below 2340 Å; the arc was water-cooled and a magnet used to deflect the discharge against the front wall of the lamp. In case the direct photochemical decomposition was to be studied, no filter was used and the arc was run without water-cooling or a magnetic field. After ten minutes' illumination the liquid air was replaced on the trap to condense out the remaining ammonia, and the pressure of the residual gases was read on the McLeod gauge. Several half-periods were taken on the quartz-fibre manometer and the value of $\sum(p\sqrt{M})$ read off the calibration plot.

In both these types of experiment appreciable quantities of residual gas were found; however, when a tartaric acid filter was used and the arc run hot with no magnetic field (in order to reverse the core of the resonance line 2537 Å), no residual gas was found.

The Experimental Results.—A summary of the experimental results is given in the table. The values of $\sum(p\sqrt{M})$ considered in relation to the pressure $\sum p$ of the residual gases are much lower, both for the sensitized and unsensitized reaction, than corresponds to the molecular weight of any gas which could have been present except hydrogen. They consequently demonstrate the presence of a considerable proportion of hydrogen in the residual gas. The mol-fraction x of hydrogen was calculated by the following equation, which assumes that the non-condensible products are hydrogen and nitrogen:

$$x = \frac{\frac{1}{\sum p}\sum(p\sqrt{M}) - \sqrt{M}_{N_2}}{\sqrt{M}_{H_2} - \sqrt{M}_{N_2}}$$

where M_{H_2} and M_{N_2} are the molecular weights of hydrogen and nitrogen, respectively.

The average mol-fraction of hydrogen, computed from eight experiments on the direct photochemical decomposition of ammonia, is 0.870, while 0.75 corresponds to the reaction $2NH_3 = 3H_2 + N_2$. This high value may indicate the formation of some condensible product, such as hydrazine, containing more nitrogen than does ammonia. However, we found that the quartz tube, when illuminated with the unfiltered radiation, gave out gas to some extent. Thus the evacuated tube developed, on illumination, a pressure amounting to several per cent of the residual gas pressure developed in the ammonia experiments. This fact renders a closer interpretation of the value 0.870 impossible.

The fact that the filtered radiation from the cooled arc produced non-condensible gases, while that from the hot arc gave none, shows that

optical excitation of the mercury vapor by the line 2537 Å is the first step in the sensitized decomposition.

The average of eleven experiments on the sensitized decomposition gave 0.699 for the mol-fraction of hydrogen in the non-condensible products. This is somewhat lower than the value 0.75 for an ammonia mixture. It is known, however, from the work of Cario and Franck[4] and others that hydrogen activated by collision with excited mercury "cleans up" on the walls of a quartz vessel. Since hydrogen is one of the decomposition products of ammonia, it may be expected to adsorb on the walls. This would leave a residual gas with a somewhat smaller mol-fraction of hydrogen than corresponds to an ammonia mixture, and thus account for our low results. We have no evidence of other products than hydrogen and nitrogen in the sensitized decomposition.

A greenish fluorescence, bright at first and dying out somewhat as the reaction proceeded, was noticed when the lamp was cooled, but not when it was hot. On observing with a direct-vision spectroscope, the fluorescence was found to be a diffuse band with a maximum around 5100 Å. On photographing with a quartz-prism spectrograph an additional diffuse band was found in the ultra-violet with a maximum around 3370 Å. Similar photographs taken with no ammonia in the tube, and one taken with about the same pressure of air in the tube, showed no bands. Visual observations were made with the tube filled with nitrogen and the green fluorescence found. Comparative observations in which two quartz tubes were placed side by side, one containing nitrogen at 3 mm. and the other ammonia at the same pressure, showed that the fluorescence in the tube containing nitrogen was markedly more intense. Similar observations with hydrogen showed no visible fluorescence. The interposition of a glass plate between the mercury arc and the quartz tubes caused the fluorescence to cease. In all these experiments the mercury was at room temperature and not distilling. These diffuse bands appear to be identical with those observed by Phillips[5] in a stream of distilling mercury vapor, excited by the line 2537 Å. Van der Lingen and Wood[6] have also excited the same bands in distilling mercury by means of a zinc spark. These bands were also noted by Wood[7] when light from a cooled mercury arc was allowed to shine on a tube containing mercury vapor at room temperature, and a few millimeters of nitrogen. He further found that this gas mixture absorbed the line 4046 Å ($2p_3 - 2s$) while fluorescing, indicating the presence of mercury atoms in the $2p_3$ state. It seems likely that ammonia as well as nitrogen can produce this band fluorescence of mercury, since in our experiments the fluorescence was visible at the very beginning of the illumination of the ammonia-mercury mixture. Contrary to the observation of Wood, we found these bands unaccompanied by the "water band" just below 3125 Å.

Summary.—It has been shown that the photochemical decomposition of ammonia at low pressure into nitrogen and hydrogen is sensitized by mercury vapor when the radiation used is from a cooled quartz mercury-arc lamp and is filtered so as to remove wave-lengths below 2340 Å. The so sensitized decomposition has been found to be accompanied by diffuse-band fluorescence in the green and ultra-violet.

[1] Kuhn, *Compt. Rend.*, **177**, 956 (1923).
[2] Coolidge, *J. Am. Chem. Soc.*, **45**, 1637 (1923).
[3] Haber and Kerschbaum, *Zt. Elektrochem.*, **20**, 296 (1924).
[4] G. Cario and J. Franck, *Zt. Physik*, **11**, 161 (1922).
[5] Phillips, *Proc. Roy. Soc.*, **89**, 44 (1913).
[6] Van der Lingen and Wood, *Astrophys. J.*, **54**, 149 (1921).
[7] Wood, *Phil. Mag.*, **50**, 774 (1925).

THE CONCENTRATION AND IDENTIFICATION OF THE ELEMENT OF ATOMIC NUMBER 61

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In making measurements of the wave-lengths of the X-ray K emission lines for the rare-earth elements,[1] very faint traces of lines corresponding to the K series of the element of atomic number 61 appeared on the plate with certain specimens of samarium (atomic number 62) and neodymium (atomic number 60). In tracing these compounds to their source a consistent effort has been made over a period of two years to increase the concentration of the element of atomic number 61. Many of the samples obtained have shown definite traces of lines where they should be expected in both the K and L X-ray wave-length regions but the photographic spectra have until now never been strong enough to permit photographic reproduction in a printed article and this was deemed necessary before the announcement of the discovery of the element was made.

In 1923, two of the authors commenced examining rare earth minerals for element 61. The minerals, which have been thoroughly investigated by employing large quantities, comprise gadolinite, xenotime (ytterspar) and monazite. These three minerals show an interesting variation in their rare earth constituents. Monazite is essentially a phosphate of the cerium group elements carrying a few per cent of the yttrium earths; gadolinite consists largely of the yttrium group with some of the cerium

group; ytterspar is just the reverse of monazite, in that it is composed of phosphates of the yttrium earths, with small amounts of those from the cerium group. The minerals were decomposed and the neodymium-samarium content separated by the usual methods of fractionation. In the case of gadolinite and monazite this Nd-Sa portion had to be put through an exhaustive fractionation before even the faintest line of 61 could be discerned. On the other hand, the ytterspar material gave faint lines after just a few crystallizations.

Ytterspar proved to be of exceptional interest owing to the fact that it was a very curious variety which carried a very much larger proportion of gadolinium and dysprosium than any other mineral met with. The solution of the rare earths, which showed only weak absorption bands of neodymium, gave upon treatment with sodium sulphate a copious precipitate of double sulphates. Since these were practically white, it was at first supposed that they consisted mainly of lanthanum and cerium. However, upon conversion to the double magnesium nitrates and fractional crystallization, it was found that the major part of the material quickly passed into the fractions more soluble than those of neodymium and samarium. These double nitrates gave an almost colorless solution showing weak dysprosium bands, indicating that the chief constituent was gadolinium.

In 1914 in an article upon terbium[2] it was stated that certain fractions coming between terbium bromate and the less soluble gadolinium salt showed some faint absorption lines attributed to neodymium. It would be concluded from this that Nd bromate was less soluble than Tb bromate. Apparently contradictory to this conclusion, J. Zernike in the winter of 1924–25 while attempting to facilitate the separation of Ho from Dy and Pr from Nd, by fractionating a mixture of the bromates of Pr, Nd, Ho, Dy and Tb, observed that terbium was less soluble than neodymium.

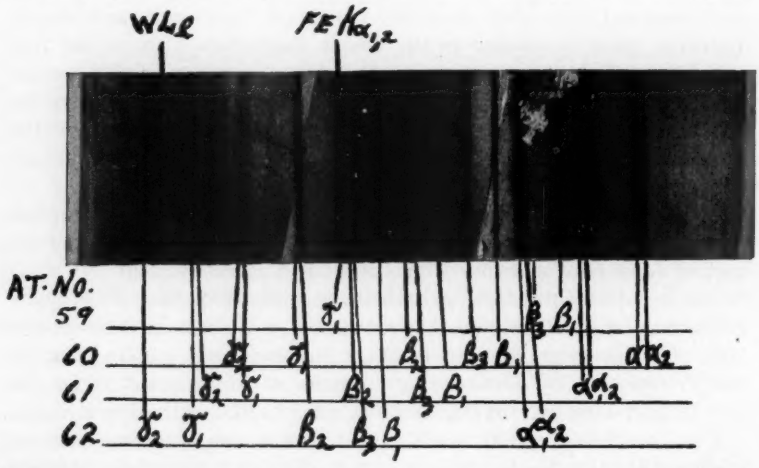
These conflicting results would indicate either that two elements giving similar absorption bands exist or that the solubility curves of the bromates of Nd and Tb cross at about ordinary temperature. The latter possibility could be easily investigated by a very careful determination of the solubility curves of these salts. Such work was undertaken and it was found that the curves of the above-mentioned compounds did not cross between zero and 45°. Neodymium bromate proved to be more soluble than terbium bromate.

A careful examination of the fractions of bromates more soluble than terbium obtained by Bissell and one of the authors, as previously mentioned, showed the complete absence of neodymium absorption bands. This would indicate that the faint absorption bands shown in addition to that of terbium in the terbium gadolinium fractions were due to element 61.

In order to confirm the presence of element 61, the minute amount of neodymium which accompanied the yttrium earths derived from a very large quantity of Brazilian monazite sand was roughly concentrated by fractionally crystallizing the bromates and lastly the double magnesium nitrates. This neodymium concentrate which was expected to carry the major portion of element 61 was then submitted to X-ray examination.

X-Ray Identification.—Figure 1 shows a reproduction of the X-ray *L* series lines obtained by means of a Siegbahn vacuum spectrograph. Three overlapping plates were taken to cover the complete region. The displacement of the lines with atomic number serves as a very positive method of identification. Besides obtaining lines in the position in which they would be expected, it is further necessary to show the impossibility of their existence due to impurities.

The target of the X-ray tube contained iron and copper, while tungsten



lines always appear due to depositions on the target from the heated filament. The powdered sample was rubbed lightly on the face of the target at the beginning of each exposure.

This sample contained the elements samarium (62), neodymium (60), praesodymium (59) and a slight amount of cerium (58) in addition to (61). The estimated amount of oxalate of element (61) was from 1 to 1.5 per cent.

While there are more than twenty *L* series lines for each element only about 7 of these are fairly strong, namely α_1, α_2, β_1, β_2, β_3, γ_1, γ_2 (Siegbahn notation). All seven of these lines for element 61 are observable on the original plates although some of them are rather faint in the reproduction. The strongest line is α_1. This lies approximately midway between the

corresponding lines for elements 60 and 62 as do all of the other lines for element 61. Using the values of Siegbahn for elements 60 and 62 the wave-lengths of the L series lines for element 61 are as follows:

α_2	α_1	β_1	β_3	β_2	γ_1	γ_2
2.289	2.279	2.078	2.038	1.952	1.799	1.725

That the a_1 line is not due to impurities may be shown. Some of the possibilities of a line in this neighborhood are V K_1^β (2.280), Cr K_1^α (2.285), Ce L_6^β (2.277) and La L_9^β (2.277). If the lines were due to the K series of any element the most prominent lines would be the $\alpha_1\alpha_2$ doublet similar to that for Fe shown in the figure. A test was made for the $\alpha_1\alpha_2$ lines of vanadium and traces of it were found, but the intensity was such that the β_1 line having an intensity much less would be difficult to observe. The L_6^β line of cerium (58) would have an intensity of about one-tenth that of the β_1 line for cerium. The β_1 line, however, is just barely discernible, therefore any of the β lines with subscripts greater than three would be too faint to appear. Further, along with the α_1 line appears the α_2 line in its proper place and intensity. While it is difficult to observe this in the reproduction, it was sufficiently strong to be measured by the comparator on the original plates. While the other lines attributed to 61 are less distinct, an analysis of the possibility of their being due to impurities, particularly in the case of β_1, β_3 and γ_2, confirms their assignment.

[1] *Physic. Rev.*, **25**, p. 197.
[2] *J. Amer. Chem. Soc.*, **36**, p. 2060.

THE DENSITY OF OXYGEN AND ITS COMPRESSIBILITY BELOW ONE ATMOSPHERE. II

BY GREGORY PAUL BAXTER AND HOWARD WARNER STARKWEATHER

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Experiments upon the normal density of oxygen have recently been reported.[1] The final result of these experiments for the weight of the normal liter, 1.42901, is subject to a correction of −0.00003, owing to the use in the calculations of an incorrect value for the force of gravity at the Coolidge Laboratory. If the true value, 980.399, is employed, the corrected result of all our earlier experiments is 1.42898, a value which is identical with the result of the last three and unquestionably the best series. This corrected value has already been used in computing the atomic weight of helium.[2]

Since two liter globes were employed in work upon the density and compressibility of helium,[2] new experiments with oxygen were made using these globes at atmospheric and also at lower pressures.

TABLE 1

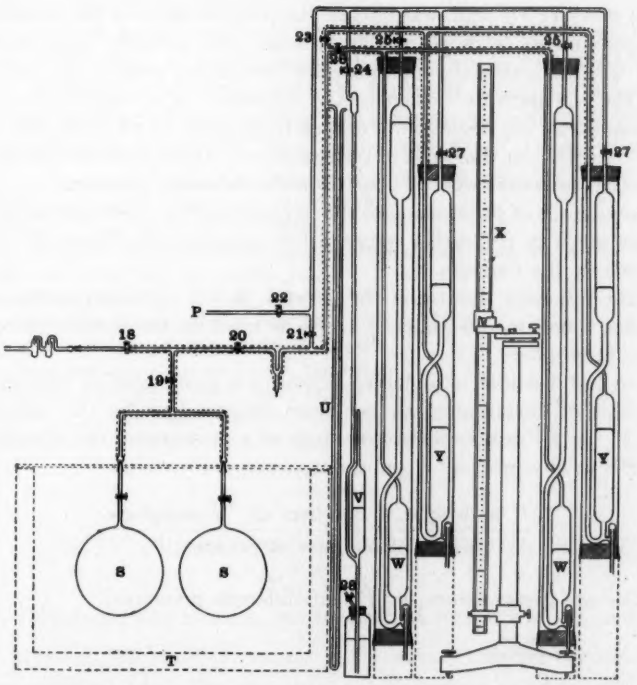
THE DENSITY OF OXYGEN

SERIES	PREPARATION OF OXYGEN	GLOBE IV 2110.95 ML.	GLOBE V 2117.64 ML.	GLOBE VI 2117.61 ML.	GLOBE VII 2117.77 ML.	AVERAGE
		0°C. and 760 mm. at sea level, Lat. 45°				
23	I	1.42895	1.42896			1.42896
24	II	1.42898	1.42897			1.42898
25	III	1.42896	1.42895			1.42896
	Average	1.42896	1.42896			1.42896
		0°C. and 570 mm. at sea level, Lat. 45°				
26	IV	1.07145	1.07148			1.07147
27	V	1.07139	1.07144			1.07142
28	VI	1.07148	1.07148			1.07148
29	VII	1.07148	1.07148			1.07148
38	X	1.07156			1.07157	1.07157
42	XI	1.07161			1.07161	1.07161
43	XI	1.07143			1.07135	1.07139
44	XII	1.07148			1.07148	1.07148
45	XII	1.07148			1.07148	1.07148
46	XII	1.07150			1.07147	1.07149
	Average	1.07149	1.07147		1.07149	1.07149
		0°C. and 380 mm. at sea level, Lat. 45°				
30	VIII	0.71407		0.71418		0.71413
31	VIII	0.71419		0.71414		0.71417
32	IX	0.71416		0.71417		0.71417
33	X	0.71407			0.71414	0.71411
34	X	0.71416			0.71422	0.71419
35	X	0.71419			0.71416	0.71418
	Average	0.71414		0.71416	0.71417	0.71415
		0°C. and 190 mm. at sea level, Lat. 45°				
36	X	0.35697			0.35701	0.35699
37	X	0.35699			0.35703	0.35701
39	XI	0.35693			0.35697	0.35695
40	XI	0.35696			0.35699	0.35698
41	XI	0.35699			0.35701	0.35700
	Average	0.35697			0.35700	0.35699

Electrolytic oxygen was purified in the apparatus described in the earlier paper and was finally liquefied and fractionally distilled before use. In the later determinations sufficient oxygen for several experiments was prepared in one operation and preserved in a liquid condition surrounded with liquid air until needed. Successive fractions from the same sample showed no evidence of dissimilarity.

The interior volumes of the globes were determined at 0°. Globes

VI and VII are actually Globe V after restandardizations made necessary by accidents. The compressibilities were found to be proportional to the pressure and in accord with the Lord Rayleigh-Moles formula. The apparatus used in filling the globes and measuring the pressure was modified to include two low-pressure as well as two high-pressure barometers, all of 40 mm. diameter, and all surrounded by water at constant temperature.



P—Lead to Gaede or mercury vapor pump. *S,S*—Density globes. *T*—Ice bath for density globes. *U*—Open arm manometer. *V*—McLeod gauge. *W,W*—High-pressure barometers. *X*—Invar meter bar. *Y,Y*—Low-pressure barometers.

The two pairs of barometers always gave essentially identical readings, if both mercury columns were read on the ascent.

The meter bar was of invar and had been calibrated at centimeter intervals by the U. S. Bureau of Standards in December, 1924. Its coefficient of linear expansion is 0.00000157 between 0° and 20°. The correction increased to a maximum of 6 microns at 81 cm. and then fell to 3 microns at 100 cm. It was graduated in half-millimeter divisions.

Pressure corrections were applied for a small difference in level between the center of the globes and the lower level of the barometers. In the low-pressure determinations the interior volumes of the globes were corrected for compressibilities of the globes themselves. This amounted to 0.36 ml. per atmosphere. In all the determinations the weight of gas was subjected to a small correction due to the compressibility of the globe and consequent variable displacement of air.

The pressure 570 mm. was outside the possible range of the barometers. The experiments recorded at this pressure were actually made either at about 620 mm. with the high-pressure barometers or at about 515 mm. with the low-pressure barometers. Allowance was then made in the calculations for the deviation of oxygen from Boyle's Law, using the coefficient -0.00092 for the range 0–1 atmosphere. In all other determinations the actual pressures were not far from the reference pressure.

Other details of purification of the oxygen and the experimental procedure are given in the earlier paper. The apparatus for filling the globes is shown in the diagram.

In the following discussion the average of our corrected earlier result and that found in this paper, 1.42897, is used for the density of oxygen at one atmosphere.

From the densities at different pressures it is possible to compute the coefficient of deviation of oxygen from Boyle's Law by the method of Guye.[3,4] If PV is assumed to be unity at a pressure of one atmosphere, then PV at any pressure P (in atmospheres) is

$$\frac{P \times \text{weight of one liter at 1 atmosphere}}{\text{weight of one liter at pressure } P}.$$

Below are given values for PV at different pressures:

PRESSURE ATMOSPHERES	PV
1	1
0.75	1.00022
0.50	1.00047
0.25	1.00071

The excess in value of PV over unity is inversely proportional to the pressure within the limit of error of the experimental evidence. If each value of PV is given equal weight, that corresponding to zero pressure is 1.00092. The coefficient of deviation from Boyle's Law per atmosphere, -0.00092, is slightly lower than that found by Lord Rayleigh,[5] -0.00094, Jaquerod and Scheuer,[6] -0.00097, Holborn and Otto,[7] -0.00099, and Kuypers and Onnes,[8] -0.00096; and slightly higher than that found by Guye and Batuecas,[9] -0.00085, and by Batuecas, Maverick

and Schlatter,[10] −0.00087; and is very near the average calculated by Pickering,[11] −0.00094.

With the density at one normal atmosphere, 1.42897, and the coefficient of deviation from Boyle's Law per atmosphere, −0.00092, the limiting value of molal volume is 22.4144 liters.

The atomic weight of helium calculated with this value from the density previously found by us, 0.17846,[2] is 4.0001, where the fourth decimal place is without especial significance.

We are very greatly indebted to the Milton Fund for Research in Harvard University, to the BACHE and WOLCOTT GIBBS FUNDS of the NATIONAL ACADEMY OF SCIENCES and to the Cyrus M. Warren Fund of the American Academy of Arts and Sciences for generous assistance, without which this work would not have been possible.

[1] Baxter and Starkweather, these PROCEEDINGS, **10**, 479 (1924).
[2] Baxter and Starkweather, *Ibid.*, **12**, 20 (1926).
[3] Guye, *J. chim. phys.*, **17**, 171 (1919).
[4] The defects of this method have recently been discussed by one of us. Baxter, *J. Amer. Chem. Soc.*, **44**, 595 (1922).
[5] Rayleigh, *Phil. Trans.*, **A204**, 367 (1905).
[6] Jaquerod and Scheuer, *Compt. rend.*, **140**, 1384 (1905).
[7] Holborn and Otto, *Z. Physik*, **10**, 367 (1922).
[8] Kuypers and Onnes, *Comm. Leiden*, No. **165**, 3 (1923).
[9] Guye and Batuecas, *J. chim. phys.*, **20**, 308 (1923).
[10] Batuecas, Maverick and Schlatter, *Ibid.*, **22**, 131 (1925).
[11] Pickering, *U. S. Bureau of Standards, Circular* **279**, 44 (1925).

THE DENSITY, COMPRESSIBILITY AND ATOMIC WEIGHT OF NITROGEN

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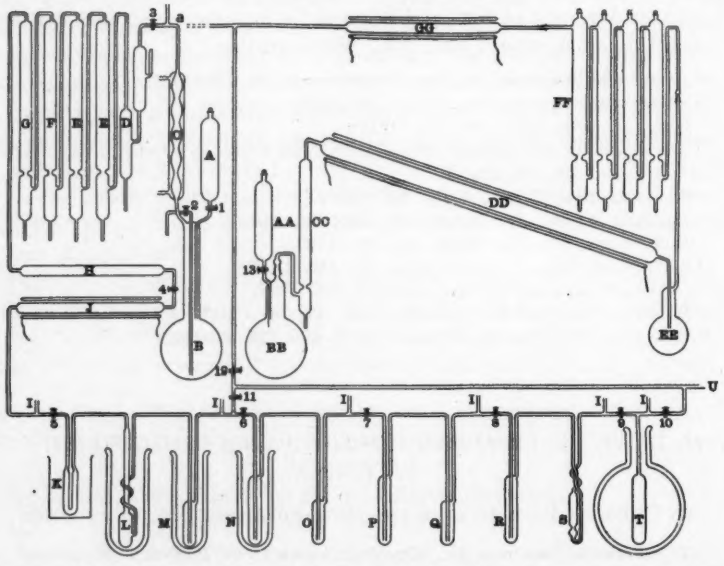
As described in the foregoing and in an earlier paper on oxygen, the density of nitrogen at different pressures has been redetermined. The gas was prepared (1) by decomposition of ammonium nitrite, and (2) by oxidation of ammonia. The apparatus for generating and purifying the nitrogen is shown in the figure.

A concentrated solution of sodium nitrite in *A* was added as desired to a concentrated solution of ammonium sulphate in *B* and the evolution of gas was controlled through the temperature. The gas was passed through the condenser *C*, over dilute sulphuric acid in the tube *D* and the towers *E*, *E*, over concentrated aqueous potassium hydroxide in the tower

F, over anhydrous potassium hydroxide in the tower *G*, over resublimed phosphorus pentoxide in the tube *H*, over pure hot nickel in the tube *J*, over dehydrated chabazite chilled with ice in *K*, and with liquid air in *L*. The purified gas was liquefied in *N*, which was surrounded with commercial liquid nitrogen under reduced pressure, after bubbling through a portion of previously liquefied nitrogen in *M*. Fractional distillation from *N* to *P*, and from *P* to *R* and from *R* to *T* followed, the most and least volatile fractions being allowed to escape through the safety tubes *I*, *I*, *I*, *I*, which dipped under mercury. In each distillation the vapor was bubbled through liquid nitrogen in *O*, *Q* and *S*, produced by liquefaction of the first portion



of gas. The final product was stored under slight excess pressure in *T* which was kept surrounded with liquid nitrogen, each specimen of gas serving for several determinations of the density.

To prepare nitrogen from ammonia, concentrated aqueous ammonia in *AA* was dropped on fused potassium hydroxide in *BB*, and the ammonia after being dried with solid potassium hydroxide in *CC* was passed over hot copper oxide in *DD*. The flask *EE* contained a small amount of sulphuric acid and served as condenser and receptacle for the water formed. The gas was dried over solid potassium hydroxide in the towers *FF* and passed over hot nickel in *GG* before entering the purifying train used for nitrogen made from ammonium nitrite at *a*.

Successive fractions of each preparation of nitrogen were used in density

determinations without bringing to light any noticeable differences. In most cases determinations at two pressures were combined by partially exhausting the density globes after they had been weighed when filled at a higher pressure and making a second set of measurements of pressure and weight. The average weight of the exhausted globes, compared with a counterpoise before and after each series of experiments was used. Usually these weights agreed within 0.05 mg. There seemed to be no evidence of loss in weight from solubility in the ice bath since the algebraic sum of all the changes with both globes in twenty-five experiments was +0.39 mg. In every experiment the observed pressure was very close to the reference pressure so that no correction for deviation from Boyle's Law was necessary.

The apparatus for filling the globes at 0° and for measuring the pressure is shown on page 701 of the preceding paper and the general details of procedure are given in the first paper on oxygen.[1]

In correcting the weight of one liter of gas to 0° and 760 mm. in latitude 45° at sea level the following constants have been used:

Density of water at 0°	0.999868
Cubical coefficient of expansion of mercury	0.0001818
Coefficient of expansion of nitrogen	0.0003670
Force of gravity at latitude 45° and sea level	980.616
Force of gravity at Coolidge Laboratory	980.399

THE DENSITY OF NITROGEN

SERIES	PREPARATION OF NITROGEN	GLOBE IV 2110.95 ML.	GLOBE VII 2117.77 ML.	AVERAGE
	0° and 760 mm. at sea level, Lat. 45°			
1	NH_4NO_2 I	1.25045	1.25049	1.25047
2	NH_4NO_2 I	1.25037	1.25043	1.25040
5	NH_4NO_2 II	1.25037	1.25031	1.25034
7	NH_4NO_2 II	1.25035	1.25038	1.25037
9	NH_4NO_2 II	1.25038	1.25036	1.25037
11	NH_4NO_2 III	1.25031	1.25036	1.25034
13	NH_4NO_2 III	1.25039	1.25031	1.25035
21	NH_3 I	1.25023	1.25028	1.25026
23	NH_3 I	1.25036	1.25036	1.25036
Average		1.25036	1.25036	1.25036
	0° and 506.67 mm. at sea level, Lat. 45°			
3	NH_4NO_2 I	0.83353	0.83351	0.83352
6	NH_4NO_2 II	0.83354	0.83355	0.83354
12	NH_4NO_2 III	0.83344	0.83347	0.83346
14	NH_4NO_2 III	0.83345	0.83346	0.83346
15	NH_4NO_2 IV	0.83347	0.83346	0.83347
17	NH_4NO_2 IV	0.83346	0.83347	0.83347
19	NH_4NO_2 IV	0.83344	0.83349	0.83347
22	NH_3 I	0.83351	0.83350	0.83351
24	NH_3 I	0.83345	0.83347	0.83346
Average		0.83348	0.83349	0.83348

SERIES	PREPARATION OF NITROGEN	GLOBE IV 2110.95 ML.	GLOBE VII 2117.77 ML.	AVERAGE
	0° and 253.33 mm. at sea level, Lat. 45°			
8	NH_4NO_2 II	0.41664	0.41671	0.41668
10	NH_4NO_2 II	0.41667	0.41671	0.41669
16	NH_4NO_2 IV	0.41662	0.41664	0.41663
18	NH_4NO_2 IV	0.41663	0.41666	0.41665
20	NH_4NO_2 IV	0.41663	0.41667	0.41665
25	NH_3 I	0.41667	0.41670	0.41669
26	NH_3 I	0.41667	0.41670	0.41669
Average		0.41665	0.41669	0.41667

The observed densities at one atmosphere are slightly lower than those found by earlier experimenters.

Rayleigh and Ramsay[2]	1.25071
Gray[3]	1.25056
Leduc[4]	1.25046
Moles[5]	1.25062
Moles and Clavera[6]	1.25042
Moles[7]	1.25051

The average densities at different pressures give the following values of PV, if the value at one atmosphere is assumed to be unity.

PRESSURE ATMOSPHERES	PV
1	1
0.666667	1.00011
0.333333	1.00028

On the assumption that the deviation from Boyle's Law is inversely proportional to the pressure, and giving the values for PV at two-thirds and one-third atmosphere equal weight, the value of PV for infinite dilution is 1.00039. By the algebraic method involving equations with two powers of P the result is 1.00051. Evidence at higher pressures seems to indicate that the isothermal at 0° for nitrogen is distinctly curved, so that the second method certainly is as likely to be applicable as the first. The average value of the coefficient of deviation from Boyle's Law between 0 and 1 atmosphere -0.00045 is almost identical with that of earlier values of this coefficient determined by others. These are given in the following table.

Rayleigh[8]	−0.00056
Chappuis[9]	−0.00043
Leduc[10]	−0.00038
Holborn and Otto[11]	−0.00043
Maverick[12]	−0.00044
Smith and Taylor[13]	−0.00061
Onnes and van Urk[14]	−0.00041
Verschoyle[15]	−0.00049
Average	−0.00047

If the atomic weight of nitrogen is calculated from the normal density and our two values of the coefficient of deviation from Boyle's Law, as well as the corresponding values for oxygen, the following results are obtained.

COEFFICIENT	ATOMIC WEIGHT OF NITROGEN
−0.00039	14.007 (5)
−0.00051	14.005 (9)
Average	14.006 (7)

Our results evidently lead to a value as low as 14.007 for the atomic weight of nitrogen. The value selected by the International Committee on Elements in 1923 is 0.001 unit higher than this.

We are very greatly indebted to the Milton Fund for Research in Harvard University, to the BACHE and WOLCOTT GIBBS FUNDS of the NATIONAL ACADEMY OF SCIENCES and to the Cyrus M. Warren Fund of the American Academy of Arts and Sciences for generous assistance, without which this work would not have been possible.

[1] Baxter and Starkweather, these PROCEEDINGS, **10**, 479 (1924).
[2] Rayleigh and Ramsay, *Proc. Roy. Soc.*, **55**, 342 (1894); **57**, 265 (1895).
[3] Gray, *J. Chem. Soc.*, **87**, 160 (1905).
[4] Leduc, *Recherches sur les gaz*, Paris (1898), p. 30.
[5] Moles, *J. chim. phys.*, **19**, 283 (1921).
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[14] Onnes and van Urk, *Comm. Leiden*, No. **169**, 43 (1924).
[15] Verschoyle, *Proc. Roy. Soc.*, **A111**, 552 (1926).

THE SYNTHESIS AND DISINTEGRATION OF ATOMS AS REVEALED BY THE PHOTOGRAPHY OF WILSON CLOUD TRACKS

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In 1923 Harkins and Ryan[1] developed and applied a procedure capable of revealing new details concerning the process of atomic disintegration. The initial discovery of the disintegration of light atoms was made by Rutherford,[2] who designed a method in which the scintillation apparatus of Crookes was so modified as to show when long range H-particles or protons are ejected from atom nuclei bombarded by alpha particles.

Harkins and Ryan considered that similar modification of the cloud track apparatus of Wilson would make it capable of exhibiting a greater part of the mechanism of disintegration.

That the photography of Wilson cloud tracks had not been used earlier for the purpose is undoubtedly due to the fact that the disintegration of atoms under bombardment is such a rare event that this method had not appeared to others as suitable. It will be shown later, too, that the change in the source of alpha rays, from the polonium applied earlier to a source such as ThC or RaC, which emits fast rays, was essential to the success of the method.

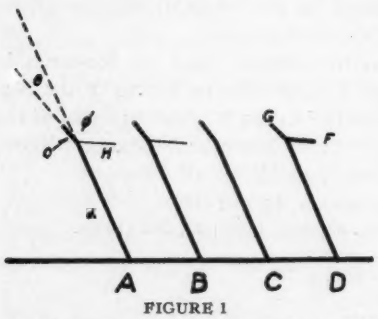

FIGURE 1

A designates the angles and rays; *B* shows the tracks as they occur in the ionization chamber in fork 1, a disintegrative-synthesis; except that the relative intensities are represented much better by *A*. *C* indicates the appearance fork 1 would exhibit provided the *H*-track were too dim to be seen; *D* indicates the characteristics of an elastic collision in which the α-particle rebounds along the track *F*. In this case there is a noticeable difference in the relations of the angles θ and ϕ.

By the use of the extraordinarily fast rays from ThC, Harkins and Ryan obtained 11,000 photographs in nitrogen and 20,000 in argon, which corresponded to about 36,000 α-tracks of 8.6 cm. range in nitrogen and 70,000 in argon. That no H-disintegrations of the ordinary type were obtained in this work was partly due to the fact that a considerable part of the work was carried out in argon, which disintegrates less often than nitrogen, and partly to a statistical accident.

Two years later Blackett,[3] by the use of the procedure applied earlier by Harkins and Ryan, obtained 22,000 photographs in nitrogen. In these eight disintegrations of nitrogen atoms to give protons were obtained and it was found that in each of these rare instances a fast alpha particle attaches itself to the nucleus of the nitrogen atom.

The purpose of this paper is to present the results obtained from 34,000 photographs not previously reported of α-ray tracks in nitrogen. These contained an average of slightly more than 12 alpha tracks each, so that about 270,000 tracks of 8.6 cm. and 145,000 tracks of 4.9 cm. range were obtained.

Figure 1 shows the arrangement in a plane of the tracks in one of the cases in which a fast α-particle attaches itself to the nucleus of a nitrogen atom. This probably forms momentarily the nucleus of an atom of fluorine, which disintegrates almost at once to give a fast H-particle or

proton, and the nucleus of a heavier atom, presumably that of oxygen of mass 17. All three tracks lie in a plane. The angle θ between the direction of the track of the α-particle and that of the H-particle is 118°. The photograph indicates that the track of the proton has only about *one-sixth* to *one-tenth* the intensity of that of the α-particle or of the oxygen atom. Figure 1*A* designates the respective tracks and angles; figure 1*B*

FIGURE 2

Two views at right angles of the non-elastic collision represented in figure 1*B*. At the left of each view an α-particle impinges upon the nucleus of a nitrogen atom and unites with it. An extremely faint straight track extends horizontally to the right of the fork and is due to a proton or H-particle emitted at high speed. This has a range of 19.6 cm. in air at 15° and 760 mm. pressure. The track of the oxygen atom which is formed slants upward to the left from the fork. The left-hand view is overexposed as a result of an irregularity in the lighting, but the track of the H-particle is easily visible on the original negative.

represents the combined synthesis and disintegration; and figure 1*C* illustrates what would be formed provided the track of the H-particle should be so faint as to be invisible. Figure 1*D* illustrates the marked difference in the appearance when the α-particle rebounds in the case of an ordinary (but extremely rare) collision. Here the dim track is not present and the angles are noticeably different. Figure 2 is a reproduction of a photograph of the disintegrative-synthesis or atomic methasesis represented by figure 1B.

Table 1 exhibits the characteristics of two abnormal forks.

TABLE 1

CHARACTERISTICS OF FORKS WHICH REPRESENT NON-ELASTIC COLLISIONS

	ϕ	θ	R_α	$V_\alpha \times 10^{-9}$	$V_p \times 10^{-9}$	R_p	$V_n \times 10^{-9}$	R_n	E_2/E_1
Fork 1	118°32′	15°21′	6.3	1.86	2.7	19.6	0.534	0.43	0.89
Fork 2		51°	ca. 7						>1.0

R = remaining range of α-particle; the distance the α-particle would have traveled beyond the point of the fork if the collision were not incurred.

V = velocity of the α-particle in cm. per second immediately before the collision.

V_p = velocity of the proton immediately after its escape from the heavy nucleus.

R_p = range of the proton, or H-particle.

V_n = velocity of the oxygen nucleus immediately after the escape of the proton.

R_n = range of the nucleus of the oxygen atom.

E_2/E_1 = kinetic energy of the H-particle plus that of the oxygen atom, divided by the kinetic energy of the α-particle immediately before impact. $1-E_2/E_1$ represents the loss of kinetic energy due to the non-elastic impact, and is presumably stored up in the oxygen nucleus of mass 17.

Fork 1 represents a case in which it seems certain that the α-particle attaches itself to the nucleus of a heavier atom, and that a proton is emitted. The appearance of the dim track is, as exactly as can be determined, just that of the track of an H-particle. It is a straight line, and, as has already been stated, has on the original film only about one-tenth the intensity of the tracks of the α-particle and the oxygen atom. Furthermore, the relation between the values of the angles θ and ϕ is not consistent with the assumption that the dim track is due to a rebound of the α-particle. Although very many of such rebounds have been photographed in this laboratory, not one of them exhibits such a dim track. If this were merely an elastic collision then the ratio of the atomic masses would be given by

$$\frac{m}{M} = \frac{14}{4} = \frac{\sin \phi}{\sin (2\theta + \phi)}.$$

By the application of this equation to fork 1 the result $3.5 = 1.7$ is obtained, and it has not been possible upon repetition of the measurements to obtain results consistent with this equation, so the collision seems to be non-elastic.

The evidence that fork 2 represents a similar disintegrative synthesis is not so good, since this collision took place near the edge of the cylinder and the tracks are too diffuse for exact measurement. However, when all reasonable allowances are made for the inaccuracies of measurement it is impossible in any case to meet the conditions of both conservation of momentum and conservation of kinetic energy. The track attributed to hydrogen is also much more dim than that of the α-particle or of the heavy nucleus. What may seem to make this case a doubtful one is

that the calculations indicate that energy is released from the nuclei and appears as kinetic energy, whereas with fork 1, and all but one of those found by Blackett kinetic, energy disappears. The minimum value of E_2/E_1 obtained from the calculations is about 1.1. There seems little doubt that with nitrogen E_2/E_1 is usually less than unity, and it may require much additional experimental work to determine with entire definiteness if in exceptional cases energy may be released from, instead of stored up in the oxygen nucleus. Whatever the significance of fork 2 may be, it seems almost certain that the collision which it represents is inelastic.

The number of disintegrations obtained in this work is only one-fourth the number obtained by Blackett in the photography of the same number of tracks, but this smaller number seems to be in much better agreement with the reports concerning such yields in the papers of Rutherford and Chadwick (see also Petterson, "Atomzertrümmerung," page 138),[4] even when allowance is made for the fact that they used much slower rays. From this point of view the larger number of disintegrations seems to be an accident, not at all remarkable when the number of events is in any case so extremely small.

Petterson and Kirsch consider that many H-tracks may be so dim as to be entirely overlooked in the photographs. They infer that short H-tracks might be more difficult to find than those of long range. Now the work in Vienna indicates that such tracks would have ranges as low as 2 cm., instead of the 18 cm. or more found thus far by the photographic method. Although we have not worked with H-particles of such a short range it would seem that they should be much brighter than the initial part of the tracks of the fast rays, and it is only the initial portion which is visible in the photographs. However, with the short rays the angle of deviation (θ) of the heavy nucleus may be so slight as to lead the observer to the conclusion that he is dealing with a straight track. Thus his attention is not called to this specific point in the picture. Nevertheless, the H-track of fork 1 is, though dim, so very distinct on the original film that it seems probable that we have not missed more than one or two such events, even if the angle θ has been so small that it has not been recognized.

The minimum velocity for the alpha particle which has thus far been recognized as giving rise to an atomic disintegration corresponds to a remaining range of 6.2 cm., so that polonium, which gives a range (at 15° and 760 mm.) of only 3.9 cm., gives α-rays of far too small an energy (5.2×10^6 volts) to have given rise to a single one of the ten disintegrations thus far discovered. The minimum amount of energy which has proved successful in causing such a disintegrative synthesis corresponds to 7 million volts. Here, as is customary, the voltage is that required to give an electron the specified energy. Thus 3.5 million volts would be necessary to give an α-particle the required velocity.

Up to the present the photographic method has not demonstrated the non-occurrence of disintegrations which give short H-rays, though the results are not favorable to their existence. The writers are engaged in a more thorough search of the film for such events.

The procedure followed in the hunt for the tracks of protons is to look over the film with great care, and also to cut out every section which shows the slightest line in any region except where the α-particle has an extremely slow speed. If a fork exhibits two bright stubs (G, F, Fig. 1D) then the existence of an H-track is not suspected. Also if no stub F is found, only an ordinary fork is indicated, provided the bright track (G, Fig. 1D) is too long to indicate an oxygen atom (in general longer than about 0.5 cm.). Forks in which no visible stub G occurs, but in which the track beyond the bend is short (Fig. 1, C), are examined with extreme care.

The measurements reported in this paper were made by a new method which may be called the "Mesh Plate Method." It consists in the establishment in the ionization chamber, which consists of a cylinder of optical glass, of a system of coördinates in space. A circular plate of yellow brass was made of such a size as to fit the cylinder with a clearance of about 0.5 mm. This was ruled on the milling machine into squares of 1 sq. mm. area by lines 0.1 mm. in diameter. These lines were then filled with tin solder and the brass surface blackened by the use of ammoniacal ammonium carbonate. This gave very uniform, silver-white lines against black on an entirely plane surface, since the surface was planed prior to the blackening.

The mesh plate is then set on the brass piston, which closes the bottom of the ionization chamber and fits inside a cylinder of brass below the glass. The mesh plate is set in an exact position by the use of a stand with three legs fitted with screws. The bottom of each leg fits a fine screw hole in the top of the piston. By the use of the fixed mirrors, used in obtaining two photographs at right angles of each set of α-tracks, two photographs, at right angles, are taken of the mesh plate. The piston is then raised a distance of 0.02 cm. and another pair of photographs taken. This operation is repeated until the top of the chamber is reached.

In order to find the position in space represented by any point in the photographs, it is only necessary to find the point in both photographs. The film which shows the point is then placed in a special apparatus such that the position of the mesh-plate pictures can be moved, while that of the tracks which contains the point remains fixed, until the readings for the point on both projections of the mesh-point picture are the same. The x- and y-coördinates of the point are given by the markings on the mesh plate (except that tenths are estimated), and the number of the mesh-plate photograph gives the z-coördinate. The use of this method of measurement eliminates errors due to the lens, which was, however, a Taylor-

Cooke-Hobson lens of the best grade. Also errors due to the glass at the top of the cylinder, and to the glass of the mirrors are eliminated. Nevertheless all of these were made optically plane in Ryerson Physical Laboratory. If the lines are too short or too highly curved, other methods should be used. The mesh-plate method has the great advantage that it allows not only for all errors due to the use of the lens but also for differences in magnification due to the use of two views at right angles.

In this work two non-elastic collisions have been found. In order to specify the number of elastic collisions it is necessary to give an arbitrary distinction between a collision on the one hand and a close approach of the alpha particle to the nucleus of the other atom. It will be considered that any fork represents a collision if the angle of deviation (ϕ) is equal to or greater than 90°, provided the impact is elastic. If the impact is inelastic the lack of elasticity is considered as evidence that a "collision" has occurred. All other forks are classed as "close approaches." Harkins and Ryan obtained about ten collisions. In the work reported here all of the collisions have been found, but have not been counted accurately. However, one-third of the film has given nine elastic collisions in about 138,000 tracks. Since the diameter of the ionization chamber is 6.3 cm., only about 0.5 of the range of the tracks of range 8.6 cm. (at 15° and 760 mm. pressure) is represented in the photographs, but almost the complete length of those of 4.8 cm. range is given. The number of elastic, but not that of the non-elastic collisions, would be increased if the whole range were shown.

According to the Hydrogen-Helium theory of Harkins and Wilson[5] the atoms have been built up largely by the aggregation of α-particles, or by the addition of smaller aggregates of protons and electrons which almost always give to the complex nucleus a composition corresponding to a certain whole number of α-particles, either with or without additional electrons. There is considerable evidence that the half-α-particle, p_2e, may also be of importance. To what extent such α-particles preserve their individuality in the complex nucleus is a moot question. Alpha particles are emitted spontaneously from the nuclei of very heavy atoms, and are now found to unite with those of light atoms. It seems probable that the emission of the proton, which has served as an indication of the addition of the α-particle, is not essential to the synthesis. Thus it is quite likely that when oxygen or carbon has been used in the chamber, the addition of an α-particle may have occurred. The great difficulty in recognizing such an event, in the absence of a very powerful magnetic or electrical field, lies in the fact that the only apparent result is the shortening of the track of the α-particle. It seems probable that the positiveness of the double charge of the α-particle is an important factor in causing the expulsion of a positively charged proton when nitrogen is bombarded.

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[1] Harkins and Ryan, *J. Amer. Chem. Soc.*, **45**, 2095 (1923).

[2] Rutherford, *Phil. Mag.*, **37**, 538 (1919).

[3] Blackett, *Proc. Roy. Soc.*, **107**, 349 (1925).

[4] Petterson and Kirsch, *Atomzertrümmerung* (1926), p. 138.

[5] Harkins and Wilson, these PROCEEDINGS, **1**, 276 (1915); *J. Amer. Chem. Soc.*, **37**, 1367 (1915); *Phil. Mag.*, **30**, 723 (1915); and later papers.

PHOTOSENSITIZED DECOMPOSITIONS BY EXCITED MERCURY ATOMS

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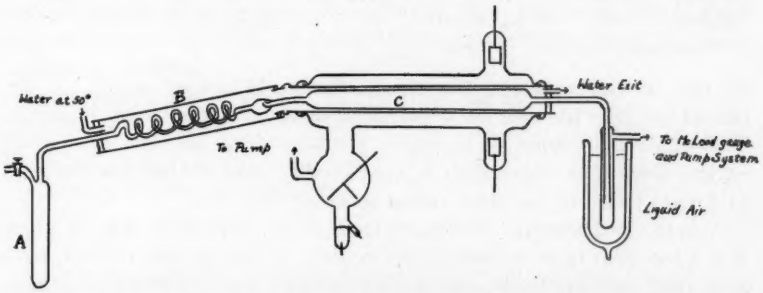
A considerable photochemical literature[1] on the subject of hydrogen atoms produced by collisions between excited mercury atoms and molecular hydrogen has served to indicate the potentialities of excited atoms for the purposes of photosensitization. The subject may be expanded in two directions, either by the use of other excited atoms or by the study of other collisions with the excited mercury. Of this latter type is the investigation by Dickinson and Sherrill,[2] arising out of their studies of the interaction of hydrogen and oxygen in presence of mercury. They have shown that the production of ozone may be secured by the action of excited mercury and oxygen even though the excitation is only that in the region $\lambda = 2536.7$ Å and, therefore, presumably inadequate for the dissociation of oxygen. They suggest, therefore, that the ozone must arise by collisions between excited oxygen molecules and normal molecules.

From our earlier work on the interaction of hydrogen and ethylene[3] we have been led to investigate the efficiency of the excited mercury atom in a wide variety of photo-decompositions and have been able to show how efficient an agent it is for such photosensitizations. It has been possible to show that water, ammonia, ethylene, methyl and ethyl alcohols, hexane, benzene, acetone, formic acid and ethyl amine are all made sensitive to decomposition by light of wave-length 2536.7 Å if the vapor exposed to the resonance radiation contains a sufficient concentration of mercury vapor. The decompositions thereby secured are, in several of the cases, many hundred-fold those secured by exposure in quartz apparatus to the whole ultra-violet spectrum of the cooled mercury arc and, in one case, the increased rate of decomposition is one-thousand fold.

The experimental arrangement is simple, as may be seen from the accompanying diagram. The material under study evaporates, at a known low vapor pressure from the reservoir A, is saturated with mercury at a given temperature in the saturator B, and then flows rapidly through the quartz vessel C, round which a cooled mercury arc of the type previously described by one of us[1] is operated. The products of reaction are then passed through two liquid air traps in which the condensible constituents are removed and the non-condensible gases (in the cases chosen for study, mainly hydrogen, nitrogen, oxygen, carbon monoxide or methane) were collected in a McLeod gauge. The rate of reaction could be measured by the rate of increase in pressure in the gauge. The gases could be analyzed in the gauge system with the aid of a platinum spiral which could be made incandescent, a copper oxide tube which could be electrically heated and a side tube which could be cooled in solid carbon dioxide or liquid air. To ascertain whether any photosensitized decomposition occurred, the



vapors in question were passed through the arc system both with and without mercury vapor. The difference in rate is attributed to the action of the mercury.

A résumé of the experimental data is presented in table 1. The second and fourth columns record the pressures in mm. of mercury of the gases entering the McLeod gauge system in 10 minutes' operation.

The data suffice to demonstrate the extraordinary efficiency of the excited mercury atom in the disruption of a variety of molecules in which the linkages involved include, C—H, N—H, O—H. The data must be regarded mainly as a qualitative index of decomposition. Nevertheless, there is a marked difference in the ease of decomposition among the examples cited. Thus, water is much less easily decomposed than methyl alcohol. Ethyl alcohol is more readily decomposed than either. Benzene is surprisingly stable. Water is more stable than ammonia. We have some evidence that the lower hydrocarbons such as methane and ethane are more stable than water. From the energy of the excited mercury atom (112 kg. cal. per gram atom) we can conclude that the energy of the

linkages N—H, O—H, C—H are all less than 112 kg. cal. in the simple hydrides of these elements. This is in accord with available thermal data. Furthermore, from the thermal data with respect to ammonia and hydrogen, $N_2 + 3H_2 = 2NH_3 + 22$ kg. cal. and $H + H = H_2 + 100$

TABLE 1

SUBSTANCE	RESIDUAL GAS WITHOUT Hg VAPOR MM.	GAS ANALYSIS	RESIDUAL GAS WITH Hg VAPOR MM.	GAS ANALYSIS	RATIO OF PHOTOSENSITIZED TO PHOTOCHEMICAL RATE
H_2O	None	.....	1.2	73% H_2; 27% O_2	....
NH_3	0.04	96% H_2; 4% N_2	7.8	89% H_2; 11% N_2	200:1
C_2H_4	None	.....	24.1	88% H_2; 12% CH_4, etc.	....
CH_3OH	0.03	.....	18.0	58% H_2; 42% CH_4 + CO	600:1
C_2H_5OH	0.7	.....	38.0	46% H_2; 50% CO + CH_4	50:1
C_6H_{14}	0.016	.....	17.0	96% H_2; 4% CH_4	1000:1
C_6H_6	0.01	.....	0.3	60% H_2; 40% CH_4	30:1
$(CH_3)_2CO$	4.7	100% CO + CH_4	9.6	100% CO + CH_4	2:1
HCOOH[a]	0.06	.....	24.4	76% CO; 24% H_2	400:1
$C_2H_5NH_2$	0.34	4% N_2; 96% H_2	20.4	96% H_2; 3.7% CH_4; 0.3% N_2	60:1

[a] 6 minutes' operation.

kg. cal., we can by setting the energy of a N—H linkage as $\gtrsim$ 112 kg. cal. set an upper limit for the dissociation energy of the nitrogen molecule. This upper limit must be $6(112) - 3(100) - 22 = 350$ kg. cal. $= 15$ volts. Even this upper limit is considerably greater than the value of 11.4 volts based on the most recent spectroscopic data.

A detailed discussion of the products of decomposition will be given in a subsequent more extensive publication. It may be pointed out, however, that, contrary to the results of Senftleben,[4] we have identified oxygen in addition to hydrogen gas in the products of decomposition of water. The hydrogen is in excess as would be anticipated from the known formation of mercuric oxide in such systems. From ethyl alcohol we have identified acetaldehyde and also its decomposition products methane and carbon monoxide, in addition to hydrogen. With benzene we have detected diphenyl, but the gaseous residue contains some hydrocarbons as well as hydrogen. With hexane the gas was nearly pure hydrogen. Formic acid yielded a larger proportion of carbon monoxide (76 per cent) than was obtained by Ramsperger and Porter[5] with a hot arc (64 per cent CO). From ethylene we have identified acetylene, hydrogen and some saturated hydrocarbons.

We desire especially to call attention to the results of the gas analyses in both the photochemical and the photosensitized decompositions of ammonia. It will be noted in each case that the percentage of hydrogen in the residue is greatly in excess of that (75 per cent) required by stoichiometric decomposition to the elements. From this excess of hydrogen we conclude that the decomposition of ammonia occurs in stages in the

first of which hydrogen is eliminated from the ammonia molecule. One such stage would be the formation of hydrazine and hydrogen from two molecules of ammonia. Our condensate, after evaporation of the ammonia, gave a residue of characteristic, somewhat irritating odor. A solution of the residue readily reduced silver nitrate solutions to metallic silver. The absence of nitrogen in the residual gas from ethylamine is also another aspect of the problem. The studies are therefore being further developed.

Summary.—The rôle of excited mercury atoms in a number of photosensitized decomposition processes has been studied.

It has been shown that water, ammonia, ethylene, methyl and ethyl alcohols, hexane and benzene, acetone, formic acid and ethylamine are all sensitized to the wave-length $\lambda = 2536.7$ Å through the agency of excited mercury.

In all cases, decomposition of the molecule occurs. Especial attention is drawn to the decomposition of ammonia, the gases from which show marked excess of hydrogen, pointing to decomposition by stages, probably through hydrazine.

[1] See Taylor, *J. Am. Chem. Soc.*, **48**, 2840 (1926), for references.
[2] *Proc. Acad. Nat. Sci.*, **12**, 175 (1926).
[3] *J. Phys. Chem.*, **29**, 842 (1925).
[4] *Z. Physik*, **37**, 529, 539 (1926).
[5] *J. Am. Chem. Soc.*, **48**, 1267 (1926).

SOME APPLICATIONS OF THE IONIZATION FORMULA

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The earlier astrophysical applications of the theory of ionization were made with the assumption that the same value of the partial electron pressure could be used throughout the atmospheres of all stars. Fowler and Milne[2] early pointed out that far lower pressures might at times be involved for certain relatively abundant atoms, and a rough estimate of the partial pressures appropriate to certain atoms in known energy states was made by the present writer, from observational data,[3] fully supporting the suggestion.

The preliminary work thus outlined, and various other considerations, indicated that conditions of great complexity govern the origin of stellar absorption lines such as we observe. Each line in a given spectrum is to be regarded as arising from an effective level that is determined by the constants of the corresponding state of the atom, and by the temperature of

the star. Furthermore, the origin of the light contained in the wings of the absorption lines must be sought at a lower level than for the line center. A quantitative beginning has, however, been made in the unravelling of the effective stratification of the atmospheres of individual stars, although the conditions that contribute to the contours of strong winged lines are still unknown.[4]

The problem of the relation between the corresponding conditions in the atmospheres of different stars is in a less satisfactory state. Although empirical methods of determining stellar luminosities have been in use for a number of years, they lack satisfactory theoretical basis and cannot be safely extrapolated to examine the conditions in the atmospheres of super-giant stars. In particular the strengthening of the absorption lines of ionized atoms, at temperatures higher than that required for their maxi-

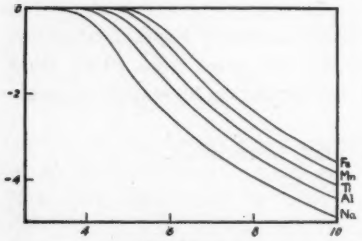

FIGURE 1

Ionization curves for the ultimate lines of the five neutral elements indicated on the right-hand margin. Ordinates are logarithms of fractional concentration; abscissae are temperatures in thousands of degrees. The curve is drawn for a partial electron pressure of 10^{-4} atmospheres.

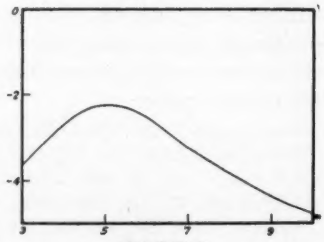

FIGURE 2

Ionization curves for subordinate lines of neutral manganese. Ordinates are logarithms of fractional concentration; abscissae are temperatures in thousands of degrees. The curve is drawn for a partial electron pressure of 10^{-4} atmospheres.

mum, appears to demand an explanation.[5] There can be no doubt that here again the problem is very involved.

Special cases of the problem of determining the relation between the temperatures and pressures of two stars have been discussed by Milne[6] and by Russell,[7] by reference to the ionization formula. The present paper is intended to furnish a graphical illustration of the principles involved in the comparison of the state of ionization of the atmospheres of stars for which both temperature and pressure differ.

The ionization diagram, as usually reproduced, is referred to a single value of the partial electron pressure, and represents the change in fractional concentration of the atomic state in question for a series of different temperatures. Two forms of the ionization curve are familiar (Figs. 1, 2), those for ultimate and subordinate lines, respectively, the former

being a special case of the latter.[8] The curves for the ultimate lines of all atoms are of identical shape, but the absolute values of the abscissae are determined by the ionization potential of the atom, increasing as this quantity decreases.

A change in the value of the partial electron pressure alters the position but not the form of the curve. Figure 3 represents the ionization curves for the subordinate lines of the atom of neutral magnesium, corresponding to a number of different partial electron pressures. The partial electron pressures represented are from 10^{-4} atmospheres downwards, since it appears from various sources of evidence[9] that pressures greater than

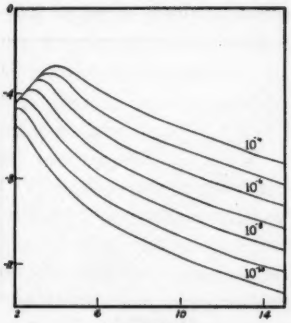

FIGURE 3

Ionization curves for subordinate lines of neutral magnesium (the solar "*b*" group), corresponding to a number of different partial electron pressures, as indicated to the right of the diagram. Ordinates are logarithms of fractional concentration; abscissae are temperatures in thousands of degrees.

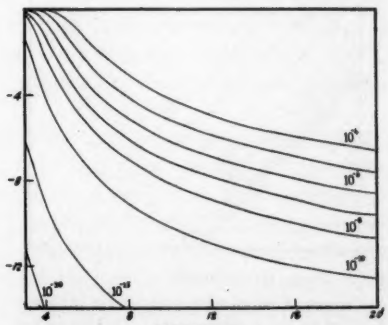

FIGURE 4

Ionization curves for ultimate lines of neutral titanium, corresponding to a number of different partial electron pressures, as indicated to the right of the diagram. Ordinates are logarithms of fractional concentrations; abscissae are temperatures in thousands of degrees.

10^{-4} atmospheres are not normally found in stellar atmospheres, although considerably smaller pressures are to be anticipated for some stars.

In the same way in which figure 1 represents curves of equal P_e for ultimate lines of the neutral atom, figure 2 shows curves of P_e for subordinate lines of the neutral or ionized atom. It is especially to be noted that the maximum value of n_r obtained decreases regularly with decreasing P_e. The fact that figure 1 actually represents a special case of figure 2 may be seen by considering that in the former the various curves, after rising to maximum (in this case the zero line) with decreasing temperature, remain tangent at that line instead of falling off again, as do the curves relating to the subordinate lines.

The change in the curve of equal P_e for ultimate lines of a neutral atom is similarly shown in figure 4 for titanium.

Another way of looking at the matter, to be employed in greater detail in the following paper, consists in plotting the curves as lines of equal temperature, the ordinate and abscissa thus becoming n_r and P_e, respectively. These curves of equal temperature are shown in figure 5 for the ultimate lines of the atom of neutral magnesium. Here again the form of the curve is naturally the same for all elements (linear for small partial pressures, tangent at the zero line for greater ones), and the position of the curve among the abscissae is determined by the ionization potential of the atom. It is to be noted that the lines of equal temperature are nearer together the hotter the gas.

Figure 6 represents the corresponding diagram for curves of equal tem-

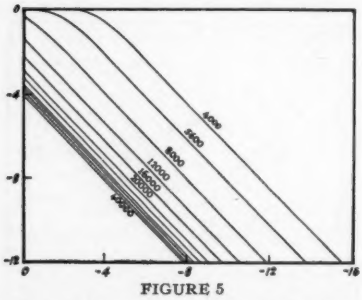

FIGURE 5

Curves of equal temperature for ultimate lines of neutral magnesium. Ordinates are logarithms of partial electron pressure in atmospheres; abscissae are logarithms of fractional concentration. The curves are drawn for temperatures from 4000° to 40,000° at intervals of 4000°, and the curve for the solar temperature of 5600° is also included.

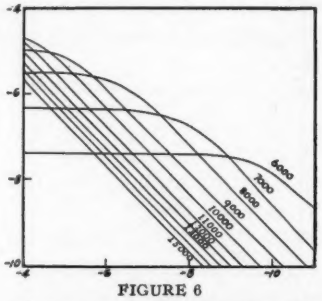

FIGURE 6

Curves of equal temperature for the subordinate line 4481 of ionized magnesium. Ordinates are logarithms of partial electron pressure in atmospheres; abscissae are logarithms of fractional concentration. The curves are drawn for temperatures from 6000° to 15,000° at intervals of 1000°.

perature for the lines of the neutral atom of magnesium ($1^3P - 1^3S$), corresponding to a prominent group of subordinate lines. The crowding of the equal-temperature curves for higher temperatures in the same way as for the ultimate lines is to be remarked. The most striking feature of the diagram is, however, the fact that each curve of equal temperature is linear at low partial pressures, and becomes tangent to a line parallel to the axis of abscissae as the partial pressure rises. The ordinate at which this tangency occurs differs for the different curves, those corresponding to high temperatures becoming tangent at greater values of n_r. On comparing figures 5 and 6 it will be seen that the former represents a special case of the latter. It is, furthermore, easy to see that figures 4 and 5, 3 and 6 represent the same results in different graphical forms.

Given such curves for any atomic state, it is possible to read off from

them directly the change in n_r corresponding to a given change in T and P_e, for the atoms in that state. In particular, if the difference of temperature between two stars is known, and the temperature of one of them, it is possible to determine the difference of partial electron pressure that would be required if the strength of the lines to which the figure relates is to remain the same for both. If we consider, for example, the stars discussed by Russell, at temperatures of 5500° and 4600°, the corresponding ratio in pressure, if the fraction of atoms of magnesium corresponding to the ultimate lines is to be the same for both, is about $10^{1.1}$, or about 13. For the same difference of temperature between two stars at 10,000° and 10,900°, the ratio in pressure, for the same equality in strength of the *b* group of magnesium, is about $10^{0.3}$, or about 2. Such ratios are of reasonable magnitude, and serve as a check on the general reliability of the method. At present it seems probable that the relation can best be used in deducing the ratios in P_e for pairs of stars with a known difference of temperature.

Considerable possibilities are opened up by the above method of comparing temperatures and pressures, if it proves to be sound. The illustrative application just made requires the equality of the fractions of magnesium in the normal state in the atmospheres of the two stars, assuming that this equality would be recognized by equal intensity of the corresponding lines in their spectra. If such an application is legitimate, it is equally legitimate to determine the ratio in pressure between the atmospheres of stars of equal temperatures, or of two known temperatures, when the ratio of the intensities of the line under consideration in the two stars is known. It is only necessary in this case to add the logarithm of the ratio in intensity to the logarithm of the ratio in pressure obtained on the assumption that the lines are of equal intensity. For a single line the procedure might well lead to results of doubtful accuracy, but if all the lines of known series relations were included in the discussion, some weight might attach to the resulting determination of the pressure ratio.

The application of the curves to the evaluation of pressure ratios is entirely dependent upon reliable measures of intensities of spectrum lines. No measures of line intensity hitherto published can be used in this connection, since they have always been made upon some arbitrary scale. The establishment of an *absolute* scale in line intensity is not a necessary adjunct, and this is fortunate as it does not seem to be possible, at the present state of stellar spectrophotometry, to establish such a scale, or even to determine the basis on which such a scale could be made, or the units in which it could best be expressed. It seems, however, to be possible to measure the fraction of the light of a given wave-length that is cut off by the presence of an absorption line. This may be expressed in terms of Ångstroms times loss of light, summed over the width of the line,

and the ratio between such quantities, measured for two stars of approximately the same temperature (whose energy background distribution may, therefore, be taken to be approximately the same) should give the most reliable intensity ratio that is at present available. Few such intensities have as yet been measured, and these, for the most part, refer to wide and diffuse lines,[10] like those of the stars of class A.

Summary.—1. The fundamental equation for ionization is applied in comparing the conditions in the atmospheres of stars that differ both in temperature and pressure.

2. The various forms of the ionization curve are reproduced, both for temperature changes at constant pressure, and for pressure changes at constant temperature. The change in the number of effective atoms for a given change of temperature and pressure can be read from either type of curve.

3. The curves for pressure change at constant temperature may be used to derive a ratio in partial electron pressure between the atmospheres of two stars of known temperatures, if the intensity ratio for the corresponding spectrum line is known.

[1] NATIONAL RESEARCH FELLOW.
[2] Fowler and Milne, *M. N. R. A. S.*, **84**, 1924 (499).
[3] Payne, *Harvard Monograph No.* **1**, 1925 (137).
[4] Milne, *M. N. R. A. S.*, **85**, 1925 (739).
[5] Payne, *Harvard Monograph No.* **1**, 1925 (140, 151).
[6] Milne, *M. N. R. A. S.*, **85**, 1925 (783).
[7] Russell, *Harv. Circ.*, **291**, 1926.
[8] Payne, *Harvard Monograph No.* **1**, 1925 (101).
[9] *Ibid.* (138).
[10] Shapley, *Harv. Bull.*, **805**, 1924; Payne and Shapley, *Harv. Reprint*, **28**, 1926.

ON THE SPECTRA OF STARS OF CLASS c F8

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The *c*-stars form a group of decided spectroscopic qualities and distinct physical properties; they have long been regarded as displaying very low values of surface gravity, and accordingly as possessing atmospheres of very low pressure. Their spectra are signalized by the strength of the lines of ionized atoms. The methods described in the previous paper will be applied in the present one to an examination of certain physical implications of the spectroscopic characteristics of the *c*-stars.

In a general study of the *c*-stars that is being undertaken by the writer, the two stars γ Cygni and δ Canis Majoris, which have very similar spectra, have been selected for special analysis. Lines of known series relations

are found in the spectra of these stars for the atoms of Fe, Fe^+, Ti, Ti^+, Ca, Ca^+, Sr, Sr^+, Sc, Sc^+, Y, Y^+. Curves of equal temperature are reproduced, for each of these atoms, in figure 1, the assumed temperature being 5600°, which is slightly lower than that for a normal star of class *F8*, to allow for the pronounced *c*-character. The curves are derived from the elementary Fowler-Milne formula, no account being taken of other

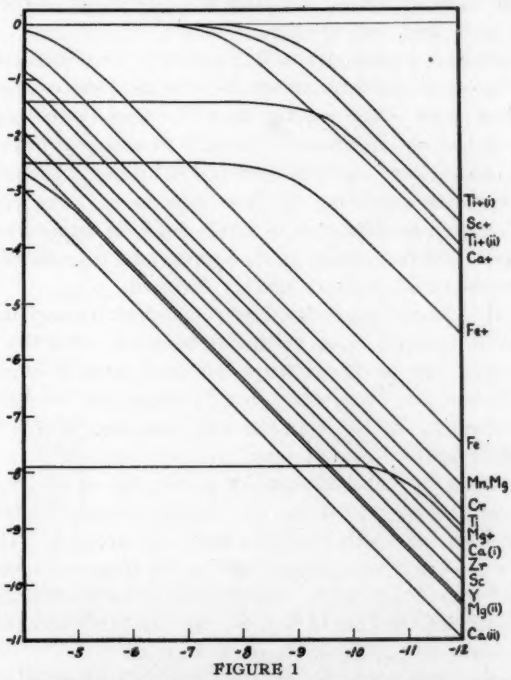

FIGURE 1

Curves of equal temperature, corresponding to the atoms indicated on the right-hand margin, drawn for T = 5600°. Ti + (i), Ca (i), and Mg represent ultimate lines; Ti + (ii), Ca (ii), and Mg (ii) represent lines of excitation potential 1.16, 1.88, and 2.67 volts, respectively.

states of the atom than the one under discussion. The corresponding correction would be determinate if the relative probabilities of the other important states could be measured or deduced.

Several marked characteristics of the spectrum of a *c*-star of class *F8* receive ready interpretation from the diagram. The excess of intensity of all the ionized lines, except those of iron, over that of the lines of the corresponding neutral atom, is at once explained by their far greater values of n_r; the excess of ionized over neutral intensity is greatest for

scandium and yttrium, next for titanium, followed closely by calcium and strontium, and is negative for iron. In the absence of accurate measures it is not possible to check this order observationally, but it may be mentioned that while the lines of the ionized atoms of yttrium, scandium, and even lanthanum and zirconium are definitely present, the lines of neutral yttrium, lanthanum and zirconium are altogether absent from the spectrum, those of neutral scandium are very faint; on the other hand the neutral iron lines are intense.

If any quantitative estimates of line intensity were available, it would be possible to obtain an estimate of the partial electron pressure in the atmospheres of stars whose spectra show the lines of the same atom in two or more stages of ionization. The ratio in apparent intensity for the neutral and ionized lines could be used to read directly from the diagram the value of P_e corresponding to that ratio in n_r. The correction for difference of n_r due to difference of level[2] could be applied to the result, and the actual relative values of P_e appropriate to each of the atomic states concerned could be thus directly deduced.

Although it is hoped that significant values of relative intensity will soon be derived[3] it seems to be worthwhile to point out a few qualitative conclusions that can be drawn from material already available. The curves for Fe and Fe^+ cross where log P_e (expressed in atmospheres) is -7. If we assume, for the moment, that the absorption efficiencies of the atoms of Fe and Fe^+ are the same, we may conclude that the value of P_e is greater than 10^{-7} atmospheres for stars, such as the sun, for which the Fe lines are stronger than the Fe^+ lines and would be less than this quantity for stars for which the Fe^+ lines were stronger. The fact that no such stars are found sets a lower limit to the observed stellar values of P_e, at least for second type stars. If the pressure for the Sun is of the order of 10^{-4} atmospheres, the lines of Sc^+, Y^+, Sr^+, and Ca^+ should be stronger than the corresponding lines for the neutral atoms.

The following table gives the values of the intensities of the strongest unblended lines of these and certain other atoms, taken from Rowland's table, which, though it does not supply intensities appropriate to the determination of intensity ratios, is a satisfactory source of intensity differences, at least for the blue and violet portions of the spectrum. The table contains, in successive columns, the atom, the wave-length of the line considered, the corresponding ionization and excitation potentials, the logarithm of the fractional concentration computed for a partial pressure of 10^{-4} atmospheres, the excess (difference between the values of log n_r for neutral and ionized atoms), and the solar intensity, taken from Rowland's table. Italicized values in the third column are estimated ionization potentials, derived from astrophysical and other evidence. They are probably of the right order; but for the present purpose an error

of two volts would not be serious, as the important quantity is the excitation potential. The ionization potential of Ti^+, marked with an asterisk, was kindly communicated by Professor Russell.

TABLE 1

ATOM	LINE	IONIZATION POTENTIAL	EXCITATION POTENTIAL	LOG n_r (P_e = 10^{-4} ATM.)	EXCESS	SOLAR INTENSITY
Mg	5183.67	7.61	2.67	$\bar{1}.6$	6.5	30
Mg^+	4481.33	14.97	8.83	$\bar{8}.1$		..
Ca	4226.73	6.09	0.00	$\bar{2}.7$	1.3	20
Ca^+	3933.66	11.82	0.00	0.0		1000
Sr	4607.34	5.67	0.00	$\bar{1}.0$	1.0	1
Sr^+	4077.71	10.98	0.00	0.0		8
Ba	5535.53	5.19	0.00	$\bar{2}.2$	1.8	000
Ba^+	4554.04	9.96	0.00	0.0		8
Sc	6305.69	5.4	0.00	$\bar{2}.0$	2.0	000
Sc^+	4314.09	*13.0*	0.60	0.0		5
Y	4128.32	5.0	0.00	$\bar{3}.7$	2.3	0
Y^+	3774.33	*12.5*	0.00	0.0		3
Ti	3998.65	6.5	0.00	$\bar{1}.0$	1.0	4
Ti^+	3088.03	13.6*	0.00	0.0		7
Ti^+	4300.05	13.6*	1.16	$\bar{2}.6$	0.4	3
V	4379.24	*6.5*	0.28	$\bar{1}.0$	1.0	4
V^+	3125.29	*12.5*	0.00	0.0		5
Cr	4254.34	6.72	0.0	$\bar{1}.2$	?	8
Cr^+	3124.97	*12.5*	?	?		4
Mn	4033.22	7.41	0.0	$\bar{1}.6$	?	7
Mn^+	3442.90	*12.5*	?	?		6
Fe	3859.91	8.05	0.0	$\bar{1}.9$	−2.4	20
Fe^+	4923.92	*12.5*	2.88	$\bar{3}.5$		5

It may be remarked that the excess of intensity of the lines of neutral chromium and neutral manganese over those of the corresponding ionized atoms indicates that these ionized lines of unassigned series relations are not ultimate lines; the excitation potential corresponding to the Cr^+ line is probably greater than that for the Mn^+ line.

The results for the solar atmosphere only confirm conclusions that are already fairly certain, and are chiefly of use as a check on the standpoint of the present paper. The corresponding results for the atmospheres of the *c*-stars are, however, of more original interest.

It is to be noticed from figure 1 that the difference of computed intensity between the ionized and neutral lines increases, as P_e is reduced, to a limit which occurs at different values of P_e for different elements, varying according to the ionization potential. As the pressure in the atmosphere of a star at a given temperature is reduced, the ionized lines will become progressively more prominent, as compared to the neutral lines, down to a limit of pressure that differs for different elements. Beyond this limit

no change in relative intensities of neutral and ionized atoms should occur, although these low pressures (which do not seem to be encountered in practice) could be detected, by comparing the line intensities for different elements, down to 10^{-10} atmospheres.

The *c*-character is displayed in varying degrees by different stars; some which show it most markedly were called *c*-stars by Miss Maury,[4] and others, in which the character is present, but less marked, were called *ac*-stars. It was natural to assume that the condition which is recognized by the *c*-character is more strongly present in the *c*-stars than in the *ac*-stars. The facts now receive the interpretation indicated above; the progression in *c*-character, of which the prominence of the ionized lines is a typical feature, is the progression that would be expected for a series of stars with decreasing pressure in the atmosphere.

Although there are degrees of *c*-character, it should not be too readily assumed that the normal giants, the *ac*-stars, and the *c*-stars form a *continuous* sequence. From the spectra this appears unlikely, but more numerous and more precise measures of line intensity are required before a definite conclusion can be reached.

The following table is in the same form as table 1, and contains corresponding data for a typical *c*-star of class *F8*, such as γ Cygni. The last

TABLE 2

ATOM	LOG π_γ	EXCESS	INTENSIFICATION
Mg	$\bar{4}.2$	−3.9	−
Mg^+	$\bar{8}.1$		
Ca	$\bar{3}.2$	2.8	+
Ca^+	0.0		
Sr	$\bar{3}.5$	2.5	+
Sr^+	0.0		
Ba	0.0	2.2	+
Ba^+	0.0		
Se	$\bar{4}.5$	3.5	+
Se^+	0.0		
Y	$\bar{4}.2$	3.8	+
Y^+	0.0		
Ti	$\bar{3}.5$	2.5	+
Ti^+	0.0	0.9	+
Ti^+	$\bar{2}.6$		
V	$\bar{3}.8$	2.2	+
V^+	0.0		
Cr	$\bar{3}.8$	?	+
Cr^+	?	?	
Mn	$\bar{2}.4$	?	−
Mn^+	?		
Fe	$\bar{1}.0$	−1.5	−
Fe^+	$\bar{3}.5$		

column, headed "Intensification," indicates the relative behavior of the lines of the ionized and neutral atoms; the plus sign indicates that the ionized lines are the stronger, and the minus sign, that the neutral lines are the more intense. The intensifications contained in this column refer not to definite pairs of lines, as in table 1, but to the general relative behavior of the neutral and ionized lines throughout the spectrum. A value of $10^{-5.5}$ atmospheres is used for P_e, instead of the value 10^{-4} atmospheres, as for the sun. This estimate is adopted from various considerations; if we assume that the masses and temperatures of the stars are equal, and that the partial electron pressures in their atmospheres are proportional to the square root of the surface gravity,[5] it provides a difference of seven and a half magnitudes between the sun and γ Cygni, a value that is certainly not in excess of the truth. The temperatures of γ Cygni and the sun are probably nearly equal; if γ Cygni is more massive than the sun, and the surface gravities are to remain in a given ratio, γ Cygni must be larger than the previous assumption required, and its total luminosity, therefore, still greater than that of the sun. A value of $10^{-5.5}$ is, therefore, *assumed* for present purposes, as the partial pressure in the atmosphere of γ Cygni; later it will be possible to invert the procedure and derive the ratio in P_e from measured differences in line intensity.

The table shows that the well-known features in the spectrum of a *c*-star can be qualitatively accounted for. The stronger line has in every case a numerically smaller value of $\log n_r$, corresponding to a greater fractional concentration of the atom concerned. Moreover, the logarithm of the ratio of ionized to neutral atoms of any one element (contained in the column headed "excess") is greater in every case than it is for the corresponding atoms in table 1, expressing numerically the greater prominence of the ionized lines in the spectrum of the *c*-star. When accurate intensity measures are available, these computed ratios can be put to a numerical test. It is to be remarked that corrections for relative abundance of any element are eliminated by the use of neutral and ionized atoms of the same element in making the numerical application.

All the above conclusions rest upon the assumption either that the formulae in question are applicable at these low pressures, or that the deviations from them affect both neutral and ionized atoms to the same extent. The absorbing efficiencies of the two types of atoms are also assumed to be equal, but this assumption could be replaced by a suitable correction, if the ratio of the absorbing efficiencies were known. The whole of the present paper is designed rather to indicate a method than to present conclusions. It should be possible, with more reliable data than are now available, to measure by means of the ionization curves the relative values of P_e, and hence the ratios in surface gravity, for pairs of stars, and to determine a quantity that is a function only of absolute magni-

tude and mass, and thus to estimate the absolute luminosities of supergiant stars.

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THE APPARITION DATES OF THE ANDROMEDE (OR BIELID) METEOR SWARMS

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In a paper reviewing Quetelet's list[1] of meteor shower dates, H. A. Newton[2] says:

"IV. *The December Periods.* There appear to be two epochs in December, each marking a distinct shower, viz.: Dec. 6th–7th, and Dec. 12th. There is no early date corresponding to the first epoch...."

By "early date" he appears to mean dates of the eighteenth century and previous. The "first epoch" seems to mean the time of the shower now known as the Andromede or Bielid shower. This shower is traced by C. P. Olivier[3] back to 1741, when it appeared in early December. Its later apparitions came earlier and earlier in the year, but with a puzzling irregularity in the rate of advance of the date.

In 1925 the question arose as to what was the proper date to suggest to a group of amateurs (the Bond Astronomical Club) for observation of these meteors. Finding the answer somewhat uncertain, the writer was led to apply to the data the method used by Newton, in the same paper, in forming his lists of the ancient shower dates of various swarms. Of his tables Newton says:

"In the following tables are given the historic dates of star showers from Quetelet's list..... They are expressed in the Gregorian calendar, and therefore represent approximately corresponding dates of the tropical year.... To express these dates in a sidereal year there is given at the same time the *corresponding* day (and fraction of a day) of 1850; that is, the time when the earth's longitude in her orbit, measured from a fixed equinox, was the same as on the day of the shower. The following formula was used in the computation.

"Let x be the number of days to be added to the recorded date, expressed in the Gregorian calendar, t the given year of the Christian era, n the

number of leap years between the given date and A.D. 1850, and l the length in days of the sidereal year. Then, evidently,

$$(1850 - t)l = x + 365(1850 - t) + N.$$

To reduce this to a form better suited for computation, observe that N is equal to the integral part of $\frac{1}{4}(1851 - t)$, minus 12, plus the correction between the Gregorian and Julian calendars for the given date. Let c be this correction, ϵ be the remainder after dividing $1851 - t$ by 4, $l = 365^d.256374$, and we obtain, by reducing,

$$x = (1850 - t) \times 0.006374 + \tfrac{1}{4}(\epsilon - 1) + 12 - c.$$

The integral part of $(1899 - t)/100$, minus the integral part of $(1999 - t)/400$, gives the value of $12 - c$.

"It will be observed that the secular variation in the value of l, the motion of the apsis of the earth's orbit, the diminution of its excentricity, and the periodic perturbations are neglected. The terms dependent on these would together very rarely amount to one-tenth of a day. The equation of the center is therefore omitted...."

Table 1 shows the data for all well-reported Andromede apparitions. The first column contains the serial number of the apparition, with suffixed a, b.... to indicate that, in the writer's judgment, the observations of different persons have to do with the same maximum. The second column contains the dates of maxima to tenths of a day, with the sign ± indicating uncertainty inherent in the report, the sign < meaning *previous to*. The column headed 1850 shows the date, Universal Time, reduced after Newton's method above. In this column ? means an uncertainty in the tenths, due either to lack of precision in the original report, or to doubt as to whether the actual maximum was seen. On account of the usual brevity of Andromede displays this latter uncertainty cannot be great. Only many observers reporting from numerous longitudes all around the globe can remove this uncertainty. Finally, a column of notes gives observer [ ], authority ( ), and other facts.

To study the relations of reduced apparition dates, the calendar years of apparitions were plotted as abscissas, 1 yr. = 1 mm., and "1850 dates" were plotted as ordinates, 1 d. = 10 mm. The plot is reproduced, to smaller scale, in figure 1.

TABLE 1

NO.	G.M.T. OF MAXIMUM	1850	NOTES [OBSERVER] (AUTHORITY)
1	1741 Nov. 25.4±, O. S.	Nov. 27.3?	St. Petersburg [Krafft]. (Quetelet; has N. S. date wrong.) Radiant?
2	1798 Dec. 7, <10 P.M.	Dec. 7.7	Near Bremen? [H. W. Brandes.] 7 per min., 480 total. Radiant?
3	1818 Nov. 19.5±	Nov. 19.7?	Gosport, Eng. (Kaemtz, from Chladni.) "Many." Radiant?

TABLE 1 (*Continued*)

NO.	G.M.T. MAXIMUM	1850	NOTES [OBSERVER] (AUTHORITY)
4	1822 Nov. 25, <10 P.M.	Nov. 25.6	English Channel [Duke of Würtemberg]. (Gruithuisen.)"...saheine bedeutende Zahl, von S. nach S.W. fallen." Probable radiant, 2 hrs. 20 min. +50°, within 10°.
5	1830 Dec. 7.5±	Dec. 7.6?	France [Raillard]. "Apparition extraordinaire." Radiant?
6*a*	1838 Dec. 6, 8.55–9.15 P.M.	Dec. 6.5?	Toulon [Flaugergues]. 42 in 20 min. Radiant in zenith, probably about 2 hrs. 04 min. + 43°.
6*b*	1838 Dec. 7, 6–7 P.M.	Dec. 7.4?	England [T. W. Webb]. 40 in 30 min., 1–3 observers. Radiant seems to have been somewhat southwest of zenith, say 23 hrs. + 40°.
6*c*	1838 Dec. 7.4±, "evening"	Dec. 7.5?	Parma [Colla]. "Un grande nombre." Radiant?
6*d*	1838 Dec. 7, 8–9 P.M.	Dec. 7.7	New Haven, Conn. [Herrick]. 93 per hour. (H. A. Newton). Radiant about 3 hrs. 15 min. + 50°.
6*e*	1838 Dec. 8.4±, "evening"	Dec. 8.5?	Brussels [Quetelet]. "Beaucoup." Radiant?
7*a*	1847 Dec. 8.5	Dec. 8.3?	Aachen [Heis]. 149 per hour. Radiant 22° + 55°.
7*b*	1847 Dec. 10.4±	Dec. 10.2?	Aachen [Heis]. 140 per hour. Radiant 22° + 55°.
8	1850 Nov. 29.4±	Nov. 29.4	Aachen [Heis]. "Ausserordentlich viele," but only 5 in Resultate. Radiant probably 15° + 62°.
9	1867 Nov. 30.4±	Nov. 30.0	Bergamo [Zezioli]. 7 meteors, 17° + 48°. (Schiaparelli.)
10	1872 Nov. 24.6	Nov. 24.0	New Haven, Conn. [Newton]. "250 before midnight," "three-fourths of them from Biela." "Very few morning and evening of the next day."
11*a*	1872 Nov. 27, 11.00–11.35	Nov. 26.7	Mauritius (Meldrum). 1160 per hour, 2678 total, between clouds. Radiant close to o and ζ Persei.
11*b*	1872 Nov. 27.4	Nov. 26.8	Europe and U. S. (A. S. Herschel, *Nature*, 7, 185–188, 1873).
12*a*	1885 Nov. 26.4±, "evening"	Nov. 25.4?	Bristol, Eng. [Denning]. 100 per hour, mostly Bielids.
12*b*	1885 Nov. 27, 8–9 P.M.	Nov. 26.0	Zi-Ka-Wei, China [Marc-Dechevrens]. 100 in 15 min. Radiant? Was not over at Nov. 28th, 4 A.M. The most easterly observation? Also observed at Mauritius, radiant about γ Andromedae, and off Cape Palmas.

TABLE 1 (*Concluded*)

NO.	G.M.T. MAXIMUM	1850	NOTES [OBSERVER] (AUTHORITY)
12*c*	1885 Nov. 27, 6.15 P.M.	Nov. 26.25	Europe and England generally, Rio and Eastern U. S. (Summary by H. A. Newton, *Am. Jour. Sci.* (3) 31, 78–79, 409–426, 1886). 75,000 per hour or more. Radiant 24.54° + 44.74°, average (Olivier).
13	1892 Nov. 23, 11.25 P.M.	Nov. 22.8	Washington (Horigan), and as far as Albuquerque. At Princeton, N. J. (C. A. Young), estimated 80–100 per min. Radiant shifted, from 1 hr. 20 min. + 40°30′ to 1 hr. 40 min. + 40°.
14	1892 Nov. 27, 8 to 11 P.M.	Nov. 26.9	Mexico, en route Torreon-New Orleans (H. A. Newton), "hopeless to count;" through railway car window. Radiant?
15	1899 Nov. 24.2	Nov. 23.6	Europe generally (Denning), 90 per hour; Eastern U. S. (Olivier). 2–3 per min. Radiant at Princeton 23° + 42¼°.

The change of 1850 date, from apparition to apparition, is due partly to changes in the position of the node where earth and meteors meet, partly to the fact that the meteor stream is not single and homogeneous, but multiple and condensed into swarms. In the figure it is seen that the points mostly lie within a slender strip narrowing downward to the right, or as the dates increase; and that many of these fall by groups quite accurately on straight lines converging to a point.

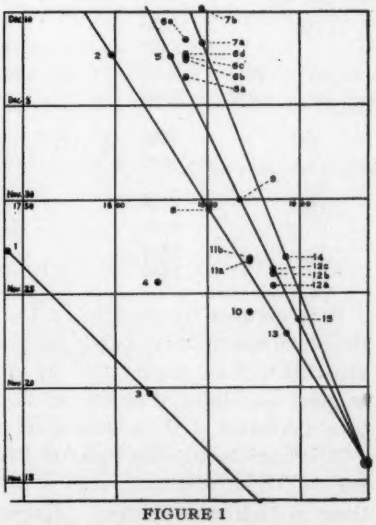

FIGURE 1

Relations of reduced apparition dates and years of apparitions.

Group (1) Apparitions (1), (3) determine a line which does not pass through or near the convergence point.

Group (2) Apparitions (2), (8), (13) fall quite accurately on a straight line, which passes between the points of apparitions (10), (11*a*), (11*b*). These last possibly all belong to one great shower of 1872, though (10) quite as readily may be taken to be a distinct apparition.

Group (3) Apparitions (5), (9), (12*c*), fall quite accurately, (15) very nearly, on a straight line. (12*c*) is the maximum for (12*a*), (12*b*), which three represent the great shower of 1885.

Group (4) The straight line determined by apparitions (7*a*) and (14) does not include (7*b*), which was no doubt part of the same shower as (7*a*), but weaker, according to Heis' data.

Apparitions (6*a*), (6*b*), (6*c*), (6*d*), (6*e*) fall between the last two lines. Apparition (4) is isolated.

The last three straight lines intersect well in a point whose coördinates are: 1935, Nov. 16.0 = 1935 Nov. 16.7, Universal Time.

The slopes of the four lines plotted indicate recessions of the nodes, freed from precession, etc., of (1) 9.9, (2) 15.8, (3) 20.5, (4) 25.3 expressed in "1850 days" per century. The rate of recession increases progressively from group to group, but not according to any obvious rule.

Supposing that these groups might have physical meaning, the intervals of the apparitions in the groups were examined for cycles or periods. Table 2 shows the results.

TABLE 2

GROUP	DATES	INTERVALS	FACTORS	INTEGER ±
(1)	1741	0	2.65 and	
	1818	77	29.06	06
(3)	1830	0	2.65 and	
	1867	37	13.96	04
	1885	55	20.76	24
	1899	69	26.04	04
(4)	1847	0	2.65 and	
	1892	45	16.98	02
(2)	1798	0	10.45 and	
	1850	52	4.976	024
	1872	74	7.082	082
	1892	94	8.995	005

It seems that for groups (1), (3), (4) the intervals have a factor 2.65, the other factors being nearly integers. The total deviation from integers amounts to 0.40, mean 0.08. In group (3), the factor 2.65 does not give as good near-integral factors as does 10.45. ($4 \times 2.65 = 10.60$.) The total deviation of these from integers is 0.111, mean 0.037.

It is curious that the mean of 2.65 and 10.45 is 6.55, the old period of Biela's comet, and their sum is 13.1, a cycle which has appeared many times in Andromede history. Examples of this cycle are shown in table 3.

From the position and convergence of the group lines, the apparitions of the Andromedes may for a few years be expected about November 16th to 19th. It is useless to hazard shower predictions for any definite year, but the convergence of the plotted lines (2), (3), (4), may indicate

some approaching meteoric event. It would be well to be prepared, for both observation and disappointment.

TABLE 3

DATES	INTERVALS	FACTORS
1741 to 1872	131	13.1 and 10.00
1741 1885	144	11.00
1741 1899	158	12.07
1798 1838	40	3.05
1798 1850	52	3.97
1822 1847	25	1.91
1847 1899	52	3.97
1867 1892	25	1.91

Appended to this paper is a bibliography of the Andromede apparitions, made up, as far as possible, from the original publications.

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LARVAL STAGES IN A PROTOZOON

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This paper deals with the Opalinidae, a family of Ciliate Infusorians, parasitic, or more properly commensal, in the recta of frogs and toads. The family comprises two subfamilies, each of which has two genera. The subfamilies and genera are all very distinct and well defined. Arranged in order of complexity and stages of evolutionary development, as I interpret the group, the classification is as follows:

CILIATA
Protociliata
Opalinidae
Protoopalininae (2 nuclei)
Protoopalina (cylindrical)
9 subgeneric groups
Zelleriella (flat)
Opalininae (4 to many nuclei)
Cepedea (cylindrical)
6 subgeneric groups
Opalina (flat)
Opalinae latae (broad, of oriental origin)
Opalinae angustae (narrow, of occidental origin)
Euciliata

The most archaic genus, dating from the Triassic period, or earlier, is *Protoopalina*. This gave rise in the early Tertiary, in South America, to the genus *Zelleriella*. From *Protoopalina* came also, probably during the Jurassic Period in the Ceylon-Seychelles-Madagascar Land Mass, *Cepedea*, by a process of multiplication of nuclei without division of the body. *Cepedea* gave rise to *Opalina*, probably in the Cretaceous period in southeastern Asia, by mere flattening of the body, giving us broad Opalinas, *Opalinae latae*. During the Pliocene, Hylas, coming from tropical America into North America, met there frogs and toads immigrant from Asia and bearing broad Opalinas of Asiatic origin. The Hylas adopted the broad Opalinas and changed them into the narrow forms, *Opalinae angustae*. This is stated here baldly—a mere summary of some of the taxonomic phases of a study occupying a couple of decades.

Dr. Wenrich, of the University of Pennsylvania, and I have noticed that Opalinids in the frog *tadpoles* are different from those in the adult host. I first observed* that the tadpoles of the Green Frog, *Rana clamitans*, have very broad *Opalinae*, almost circular, but that the oldest tadpoles, ready for metamorphosis, I later found to show these *Opalinae* as narrow forms. The narrow *Opalina* passes through a broad stage recalling its ancestral condition, thus confirming, in part, my taxonomic hypothesis.

But the chain of larval stages is longer than this. In tadpoles of *Rana clamitans* the very young *Opalinids* are elongated, cylindrical and have two nuclei. They are little Protoopalinas in appearance. They live and divide in this condition for several weeks. Later they increase the number of their nuclei by nuclear division unaccompanied by body division, becoming thus little Cepedeas. In this *Cepedea* condition they live and multiply for some weeks. Later they flatten to form very broad, circular Opalinas, the flattening beginning at the anterior end and progressing

backward. Finally, after several weeks more, the broad forms change into true *Opalinae angustae*. This species of *Opalina*, found thus far only in tadpoles of *Rana clamitans*, I named provisionally *Opalina larvarum*, pointing out at the time indications that, in spite of its very broad form, it should be classed with the *Opalinae angustae*, though at that time the narrow, final stage, in very old tadpoles, had not been seen. The name should now be made permanent, instead of provisional.

In its life subsequent to conjugation *Opalina larvarum* passes through, as noted, a well-defined series of larval stages, each stage corresponding to an ancestral genus or subgenus, being in order of development—first *Protoopalina*, then *Cepedea*, then an *Opalina lata* and ultimately reaching its definitive *Opalina angusta* condition.

In the tadpole of *Rana catesbeiana*, our northern Bull Frog, a different course is taken. The young forms of this still unnamed Opalinid are *Protoopalina*-like. These flatten to become Zelleriellas, and later increase the number of their nuclei, becoming Opalinas, first of the broad form, then of the narrow.

This all tends to confirm my interpretation of the phyletic taxonomy—the several genera and subgenera probably having evolved in the order shown in the taxonomic chart, from *Protoopalinae* to *Opalinae angustae*. The phylogeny and taxonomy were postulated as in the chart years before this recent confirmation was found in the development of the *Opalinae* in Green Frog and Bull Frog tadpoles.

These progressive stages in development, each retained for a period of some weeks, may properly be called larval stages. The fact that division, asexual reproduction, is occurring in each of these stages makes them no less larvae. Of course the recapitulative nature of these larval stages is of especial interest, helping us interpret the phyletic history of this the most archaic family of Ciliate Infusorians. The several larval stages in the life history of *Opalina* are not due to divergent adaptation to special environmental conditions, but are purely phyletic, repetitive of the ancestral history. Few, if any, finer examples of a series of developmental stages repetitive of ancestral history are found among plants or animals, either metazoan or protozoan.

Larval stages are far from unknown among Protozoa. One of the most striking examples is seen in the Suctoria, a group of Ciliata lying at the other end of the class from the archaic Opalinidae. The Suctoria, in the course of their development, have a true Ciliate stage, indicating that they have arisen from true Ciliates. This would hardly have been suspected without knowledge of the character of their larvae.

There are a number of implications in the life cycle of the higher Opalinidae, interesting to follow out, but these will be left for a fuller paper. The whole subject could be discussed in terms of the interworking of at

least four inherent trends in mutation present in the family (tendencies (*a*) to postponing the division of the body after nuclear division, (*b*) to delay in completing mitotic division of the nucleus when once begun, (*c*) to flattening of the body, (*d*) to elongation of the body). By their conjunction or disjunction in inheritance these four trends give rise to the several subfamilies and genera and to some of the subgenera mentioned in the taxonomic chart. But this treatment of a phylogeny as a shuffling of trends is too unfamiliar for condensed discussion.

* Metcalf—The Opalinid Ciliate Infusorians, Bulletin 120, United States National Museum, pp. 244–246.

TRANSFORMATIONS OF MANIFOLDS WITH A BOUNDARY

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1. The object of this note is to outline the extension of the coincidence and fixed point formulas which I have already obtained for transformations of manifolds without boundary.[1] The notations will be those of the papers just quoted.

By means of matrices we shall first greatly simplify the formulas already given. Any matrix of elements α_{ij} or α_{ij} will be denoted by α (i.e., like the elements with positional indices dropped), its transverse by α' and if it is square the sum of the main diagonal elements will be called the *trace*. We also set

$$C_\mu \equiv \| (\gamma^i_\mu \cdot \gamma^i_{n-\mu}) \|$$

where the γ's are cycles of fundamental sets for the operation $\asymp$, C_μ is square of order R_μ and its determinant ± 1 (Veblen). Then the first formula, p. 43, of the *Transactions* paper gives at once:

$$(-1)^{\mu(n+1)} C_\mu \epsilon_{n-\mu} = \alpha_\mu; \quad \epsilon_{n-\mu} = (-1)^{\mu(n+1)} C_\mu^{-1} \alpha_\mu. \qquad (1)$$

By applying (53.3), loc. cit., to the first formula on p. 47, replacing the δ's by the γ's, then making use of (1) and of the fact that transposed matrices have equal traces we find:

$$(\Gamma_n \cdot \Gamma'_n) = \Sigma(-1)^\mu \text{ trace } C_\mu \beta'_{n-\mu} C_\mu^{-1} \alpha_\mu \qquad (2)$$

whose form is as advantageous as it is condensed.

2. We now consider a manifold M_n with a boundary F_{n-1}. It is found necessary to demand that F_{n-1} be itself a manifold and that if $\overline{M}_n$ is a copy of M_n, when we match the corresponding boundary points then $V_n = M_n - \overline{M}_n$ is also a manifold (without boundary, of course). It amounts to certain homogeneity requirements along F_{n-1}.

Manageable fundamental sets for the operation $\approx$ and each dimensionality are found by associating four types of cycles: (*a*) Cycles of M_n independent of boundary cycles. (*b*) Cycles of M_n dependent upon boundary cycles. (*c*) Skew-symmetric cycles (difference between a complex on M_n and its $\overline{M}_n$ image) that meet F_{n-1} in cycles ≈ 0. (*d*) Skew-symmetric cycles that meet F_{n-1} in non-bounding cycles. It is found convenient to take in the sets for μ cycles the types in the order: $a\ c\ d\ b$ when $\mu < {}^1/_2 n$, $a\ c\ b\ d$ when $\mu \geqq {}^1/_2 n$. Then by a proper choice of the cycles of each type C_μ will have the form

$$\left\| \begin{array}{cc|c} A & 1 & 0 \\ \pm 1 & 0 & \\ \hline 0 & & B \end{array} \right\|, \quad B = \left\| \begin{array}{cc} 1 & 0 \\ & \\ 0 & 1 \end{array} \right\|, \quad \left\| \begin{array}{cc} 0 & 1 \\ & \\ 1 & 0 \end{array} \right\|, \quad \left\| \begin{array}{cc} 0 & 1 \\ & \\ -1 & 0 \end{array} \right\|.$$

The terms are all matrices, the units corresponding to unit-matrices of various orders. For $\mu \neq {}^1/_2 n$ the sign is $+$ and B is of the first form. For $\mu = {}^1/_2 n$ even, the sign is also $+$ and B of the second form, while for $\mu = {}^1/_2 n$ odd the sign is $-$ and B of the third form. The diagonal matrices are square and the sequence of their orders is the same for μ and $n - \mu$. This implies very interesting duality theorems. Let namely r_μ^i be the number of cycles in each group (in the order named), then:

$$r_\mu^1 = r_{n-\mu}^1 = r_\mu^3 = r_{n-\mu}^3; \quad r_\mu^2 = r_{n-\mu}^4.$$

3. The transformations are defined this time by means of complexes K_n on $M_n \times M_n'$ with their boundary on that of the product manifold. However, if T, T' are two transformations, K_n, K_n' the related complexes, the index $(K_n . K_n')$ is not a class-invariant unless T and T' are rather special. In the last analysis it is found advisable to choose them so that T may be approximated as closely as desired by a transformation that transforms M_n into the interior of itself and T' by one that assigns no transform to the boundary.

T, T', being so chosen, we associate with them transformations U, U' of V_n defined thus: Let $\bar{A}$ be the $\overline{M}_n$ image of a point A of M_n. Then $U\bar{A} = UA = TA$. Let also A' be any TA, then the pairs $A', \bar{A}'$, constitute all points $U'A$. If Γ_n, Γ_n' are the cycles which represent U, U' on $V_n \times V_n'$, then $(K_n . K_n') = (\Gamma_n . \Gamma_n')$. Everything is then reduced to a coincidence problem for V_n.

4. As regards U and U' what matters chiefly is that the U transform of a skew-symmetric cycle is ≈ 0, while the U' transform of any cycle is skew-symmetric. Hence the corresponding α's and β's can be exhibited as very simple matrices similar to C_μ. When this is done and (2) applied it is found that

$$(K_n . K_n') = \Sigma(-1)^\mu \text{ trace } p_{n-\mu}' P_\mu. \qquad (3)$$

Here P_μ is the transformation matrix for T and the cycles of types (a) and (b), that is, for those of M_n itself; $p_{n-\mu}$ is the transformation matrix for U' and the skew-symmetric cycles. The fixed-points formula is obtained by making in (3) all p's equal to one for T, all P's equal to one for T'. Transformations of the class of the identity are special cases of either types considered, and strikingly enough the number of signed fixed points is found again to be equal to the Euler characteristic.[2]

Very similar results can be obtained for transformations of one M_n into another.

[1] These PROCEEDINGS, **2**, 90 (1923); **11**, 290 (1925); *Trans. Amer. Math. Soc.*, **28**, 1 (1926).

[2] See these PROCEEDINGS, **11**, 290 (1925); also Hopf, *Math. Ann.*, **96**, 225 (1926).

A POSSIBLE WAY TO DISCUSS THE FUNDAMENTAL PRINCIPLES OF RELATIVITY

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Let us consider, as the element of the space-time, the point A_i, at which is installed a clock indicating a variable quantity t_i. In order to denote the different points and the corresponding readings of the clocks installed at these points we will use different values of the index i.

Suppose an electromagnetic signal were sent from A_i at the moment t_i indicated by its clock. We denote by means of the symbol $[t_i]_j^i$, the moment of reception of this signal at the point A_j recorded by the clock installed at A_j. It is evident that

$$[t_i]_j^i = \text{function of } t_i. \tag{1}$$

We will not make any assumption regarding the physical nature of the clocks. However, each variable t_i should, necessarily, increase (or decrease) continually in the course of time.

The knowledge of the functions of the type (1), given for all possible combinations of indices i and j, will supply us the complete idea of relative positions of the points A_i in the space-time.[2]

Since no assumption has been made about physical nature of the clocks, we have right to transform—if needed—a variable t_i into another variable θ_i. We can write

$$t_i = \varphi_i(\theta_i) \tag{2}$$

or, inversely

$$\theta_i = \psi_i(t_i).$$

Consequently, we have to denote the value of θ_j, corresponding to the moment of receiving at the point A_j of the electromagnetic signal sent from A_i at the moment t_i by

$$\psi_j([\varphi_i(\theta_i)]^i_j).$$

Let us consider four points, A_0, A_1, A_2, A_3. We have twelve functions

$$[t_i]^i_j;\ i \neq j;\ i,j = 0,\ 1,\ 2,\ 3. \tag{3}$$

If these functions admit such a transformation

$$\begin{aligned} t_i &= \varphi_i(t) \qquad i = 0, 1, 2, 3 \\ t &= \psi_i(t_i) \qquad \varphi_i(\psi_i(t_i)) = t_i \end{aligned} \tag{4}$$

that

$$\begin{aligned} \psi_j([\varphi_i(t)]^i_j) &= \psi_i([\varphi_j(t)]^j_i) = t + \frac{a_{ij}}{c} \\ i, j &= 0, 1, 2, 3 \qquad i \neq j \\ a_{ij} &= \text{const.} \qquad c = \text{const.} \end{aligned} \tag{5}$$

then we say that the points A_0, A_1, A_2, A_3 form a *rigid system*, which we denote, as a whole, by S. Let us call t, the uniform time of the system S; a_{ij}, distance between A_i and A_j; c, the velocity of light: this is an arbitrary factor. The value of c determines the relation between the units of time and length. [It should be noticed that, in general, the equations (4) may or may not be satisfied. We suppose that they are satisfied.]

Let A_i be any point. The functions

$$[t_i]^i_0,\ [t_i]^i_1,\ [t_i]^i_2,\ [t_i]^i_3 \tag{6}$$

determine the position of the world-line A_1 in the space-time relative to A_0, A_1, A_2, A_3.

In further, we need to introduce some axioms.

Axiom I. If the functions (6) admit such a transformation

$$t_i = \varphi_i(t) \qquad t = \psi_i(t_i) \tag{7}$$

that any three equations of the system

$$\begin{aligned} \psi_0([\varphi_i(t)]^i_0) &= t + \frac{a_{i0}}{c}, \qquad \psi_1([\varphi_i(t)]^i_1) = t + \frac{a_{i1}}{c} \\ \psi_2([\varphi_i(t)]^i_2) &= t + \frac{a_{i2}}{c}, \qquad \psi_3([\varphi_i(t)]^i_3) = t + \frac{a_{i3}}{c} \end{aligned} \tag{8}$$

(a_{i0}, a_{i1}, a_{i2}, a_{i3} being constants) are satisfied, then the remaining fourth equation will be automatically satisfied. In this case we say that the point A_i belongs to the rigid system S.

The quantities a_{i0}, a_{i1}, a_{i2}, a_{i3} are not independent among themselves: they satisfy an equation

$$F(a_{i0}^2,\ a_{i1}^2,\ a_{i2}^2,\ a_{i3}^2) = 0 \tag{9}$$

the form of which depends upon the kind of geometry used. Hence we can express a_{i0}, a_{i1}, a_{i2}, a_{i3} as the functions of three parameters U_i, V_i, W_i, which are called the coördinates of the point A_i.

$$\begin{aligned} a_{i0}^2 &= f_0(U_i,\ V_i,\ W_i), & a_{i1}^2 &= f_1(U_i,\ V_i,\ W_i) \\ a_{i2}^2 &= f_2(U_i,\ V_i,\ W_i), & a_{i3}^2 &= f_3(U_i,\ V_i,\ W_i) \end{aligned} \tag{10}$$

Axiom II. If A_i and A_j are two points, belonging to the rigid system S, then the equations

$$\psi_j([\varphi_i(t)]_j^i) = \psi_i([\varphi_j(t)]_i^j) = t + \frac{a_{ij}}{c}, \qquad a_{ij} = \text{const.} < 0 \tag{11}$$

can be satisfied by a definite value of a_{ij}, which is determined from an equation

$$a_{ij}^2 = S(U_i,\ V_i,\ W_i,\ U_j,\ V_j,\ W_j) \tag{12}$$

symmetrical in respect to the indices i and j.

The rigid system S can contain any number of points. We call the four points A_0, A_1, A_2, A_3, the basis of the rigid system. From the axiom II, it follows immediately that for any four points belonging to the rigid system S the condition (5) is satisfied. Hence any four points in the rigid system can be taken as its basis.

The equation (12) may be considered as generalization of the equation (9). Indeed, put into (12) successively $j = 0, 1, 2, 3$

$$\begin{aligned} a_{i0}^2 &= S(U_i,\ V_i,\ W_i,\ U_0,\ V_0,\ W_0), & a_{i1}^2 &= S(U_i,\ V_i,\ W_i,\ U_1,\ V_1,\ W_1) \\ a_{i2}^2 &= S(U_i,\ V_i,\ W_i,\ U_2,\ V_2,\ W_2), & a_{i3}^2 &= S(U_i,\ V_i,\ W_i,\ U_3,\ V_3,\ W_3) \end{aligned} \tag{13}$$

Since the coördinates of A_0, A_1, A_2, A_3 are known, we can obtain from (13), by eliminating U_i, V_i, W_i, the relationship involving a_{i0}^2, a_{i1}^2, a_{i2}^2, a_{i3}^2 which is identical to the equation (9).

Let us suppose that the equations (8) are not satisfied by given functions (6). In this case we will regard the quantities a_{i0}, a_{i1}, a_{i2}, a_{i3} and U_i, V_i, W_i as the functions of t. Combining the equations (8) and (10), we write

$$\psi_0([\varphi_i(t)]_0^i) = t + \frac{1}{c}\sqrt{f_0(U_i,\ V_i,\ W_i)}$$

$$\begin{aligned}\psi_1([\varphi_i(t)]_1^i) &= t + \frac{1}{c}\sqrt{f_1(U_i, V_i, W_i)}\\ \psi_2([\varphi_i(t)]_2^i) &= t + \frac{1}{c}\sqrt{f_2(U_i, V_i, W_i)}\\ \psi_3([\varphi_i(t)]_3^i) &= t + \frac{1}{c}\sqrt{f_3(U_i, V_i, W_i)}\end{aligned} \tag{14}$$

Since the functions $[t_i]_0^i$, $[t_i]_1^i$, $[t_i]_2^i$, $[t_i]_3^i$ are given, we can determine from (14) the following unknown functions:

$$\begin{aligned} t_i &= \varphi_i(t), \qquad t = \psi_i(t_i), \qquad \psi_i(\varphi_i(t)) = t\\ U_i &= U_i(t), \qquad V_i = V_i(t), \qquad W_i = W_i(t).\end{aligned} \tag{15}$$

We say, that the point A_i *moves* relatively to the system S; the last three equations (15) represent the law of its motion.

We can define the distance a_{ij} between two moving points A_i, A_j by putting their coördinates, expressed in terms of t, into equation (12). a_{ij} will be function of t.

Suppose an electromagnetic signal was sent from the point A_i at the time t. The corresponding reading of the clock, installed at A_i, is $\varphi_i(t)$. This signal will be received at the point A_j in the moment $[\varphi_i(t)]_j^i$, as recorded by the clock installed at A_j. This moment, when expressed in the uniform time t, is $\psi_j([\varphi_i(t)]_j^i)$.

The difference

$$\psi_j([\varphi_i(t)]_j^i) - t = \tau_{ij} \tag{16}$$

means the interval of time required for an electromagnetic wave to pass from A_i to A_j. This difference multiplied by c means the distance between A_i at the moment t and A_j at the moment $t + \tau_{ij} = \psi_j([\varphi_i(t)]_j^i)$. Using the equation (12), we can write:

$$c^2\tau_{ij}^2 = S(U_i(t), V_i(t), W_i(t), U_j(t + \tau_{ij}), V_j(t + \tau_{ij}), W_j(t + \tau_{ij})). \tag{17}$$

This equation should be valid for all possible combinations of indices i and j.

Suppose the points A_l, A_m, A_n, ...which do not belong to the rigid system S, form another rigid system S'. In order to distinguish the two systems we will use accents ($'$) in all notations referred to the system S'.

According to the definition of a rigid system the readings of all clocks, installed at the different points of system, can be expressed as the functions of one variable t'

$$t_i = \varphi_i'(t'), \qquad t' = \psi_i'(t_i) \tag{18}$$

in such a way that

$$\psi_m'([\varphi_i'(t')]_m^l) = \psi_l'([\varphi_m'(t')]_l^m) = t' + \frac{a'_{lm}}{c}, \qquad a'_{lm} = \text{const.} \tag{19}$$

We will use the same value of c in both systems.

Using the general property of the functions ψ_l' and φ_l', namely, $\psi_l'(\psi_l'(t')) = t'$, we can put the equations (19) into (16)

$$\tau_{lm} = \psi_m(\varphi_m'(\psi_m'[\varphi_l'(\psi_l'(\varphi_l(t)))]_m^l)) - t$$

$$= \psi_m(\varphi_m'(\psi_l'(\varphi_l(t) + \frac{a_{lm}'}{c}))) - t.$$

In order to simplify this result we can assume without making any restriction upon the problem that all clocks in the system S' indicate directly the uniform time t' of this system, i.e., that

$$t_i' = t' \qquad \varphi_i'(t') = t_i' \qquad \psi_i'(t_i) = t_i.$$

We finally obtain:

$$\tau_{em} = \psi_m\left(\varphi_l(t) + \frac{a_{lm}'}{c}\right) - t$$

$$c^2\tau_{lm}^2 = S(U_l(t), V_l(t), W_l(t), U_m(t + \tau_{lm}), V_m(t + \tau_{lm}), W_m(t + \tau_{lm})). \quad (20)$$

These equations should be valid for all points belonging to the system S'. If the geometric law, which expresses the distance between two points in terms of their coördinates, is given, we can determine from (20) the law of motion

$$U_l(t), \qquad V_l(t), \qquad W_l(t)$$

for every point of the system S' relatively to the system S and also express the reading of every clock in terms of t.

Example.—The motion of a rigid system relative to another in Euclidean space.

Let U_l, V_l, W_l be rectangular Cartesian coördinates. The fundamental geometric law (12) becomes

$$a_{lm}^2 = (U_l - U_m)^2 + (V_l - V_m)^2 + (W_l - W_m)^2.$$

The equations (20) become

$$c^2\tau_{lm}^2 = (U_l(t) - U_m(t + \tau_{lm}))^2 + V_l(t) - V_m(t + \tau_{lm}))^2 + (W_l(t) - W_l(t + \tau_{lm}))^2$$

$$\tau_{lm} = \psi_m\left(\varphi_l(t) + \frac{a_{lm}'}{c}\right) - t. \quad (21)$$

In order to solve these equations we try to satisfy them by assuming the law of motion of the form

$$U_l(t) = \overline{U}_l + pt, \qquad V_l(t) = \overline{V}_l + qt,$$
$$W_l(t) = \overline{W}_l + rt, \qquad v^2 = p^2 + q^2 + r^2.$$

Denoting by α_{lm} the angle between the direction of motion and the straight line $\overline{A_lA_m}$ we write

$$c^2\tau_{lm}^2 = a_{lm}^2 + 2\tau_{lm}\, a_{lm}v \cos \alpha_{lm} + \tau_{lm}^2 v^2. \tag{22}$$

Interchanging the indices, we obtain (since $\cos \alpha_{lm} = -\cos \alpha_{ml}$)

$$c^2\tau_{ml} = a_{lm} - 2\tau_{ml}\, a_{lm}v \cos \alpha_{lm} + \tau_{lm}^2 v^2. \tag{23}$$

Subtracting and adding the equations (22) and (23) we get:

$$\begin{aligned} \tau_{lm} - \tau_{ml} &= 2\frac{v}{c^2 - v^2} a_{lm} \cos \alpha_{lm} \\ (c^2 - v^2)(\tau_{lm}^2 + \tau_{ml}^2) &= 2a_{lm}^2\left(1 + \frac{2v^2}{c^2 - v^2}\cos^2\alpha_{lm}\right). \end{aligned} \tag{24}$$

Let us assume that the functions φ_l and ψ_l are of the form

$$\varphi_l(t) = \eta_l + zt, \qquad \psi_l(t_l) = -\frac{\eta_l}{z} + \frac{1}{z}\, t_l.$$

We easily find

$$\tau_{lm} = \frac{1}{z}(\eta_l - \eta_m) + \frac{a'_{lm}}{cz}, \qquad \tau_{ml} = \frac{1}{z}(\eta_m - \eta_l) + \frac{a'_{lm}}{cz}. \tag{25}$$

After putting (25) into (24) we get

$$\begin{aligned} \eta_l - \eta_m &= z \cdot \frac{v}{c^2 - v^2} a_{lm} \cos \alpha_{lm} \\ a'^2_{lm}\left(1 - \frac{v^2}{c^2}\right) &= z^2 a_{lm}^2 \left(1 + \frac{v^2}{c^2 - v^2}\cos^2\alpha_{lm}\right). \end{aligned} \tag{26}$$

We put

$$z = \sqrt{1 - \frac{v^2}{c^2}}$$

Finally

$$\begin{aligned} a_{lm} &= a'_{lm} \frac{\sqrt{1 - \frac{v^2}{c^2}}}{\sqrt{1 - \frac{v^2}{c^2}\sin^2 \alpha_{lm}}} \\ t'_l &= \eta_m + \frac{v}{c} \cdot \frac{a_{lm}}{\sqrt{c^2 - v^2}} \cos \alpha_{lm} + \sqrt{1 - \frac{v^2}{c^2}}\, t. \end{aligned} \tag{27}$$

The formulae (27) represent Lorentzian transformations in general form.

We see that Euclidean space admits the law of uniform translation of a rigid system relative to another. The moving system undergoes contraction in the direction of motion. This result is, apparently, the only solution of the equations (21).

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[2] A point A_t for all values of t_t may be considered as a world line in four-dimensional space-time.

CONCERNING PATHS THAT DO NOT SEPARATE A GIVEN CONTINUOUS CURVE

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In this paper it will be shown that, in space of two dimensions, every two points that do not belong to a given continuous curve may be joined by a simple continuous arc that does not disconnect that curve. First, certain auxiliary theorems will be established or, in certain cases, stated without proof.

THEOREM 1. *If M is a continuous curve there do not exist two distinct bounded complementary domains of M with the same outer*[1] *boundary.*

Theorem 1 may be proved with the aid of propositions, concerning the outer boundary of one domain with respect to another, given in footnote 4, on page 475, of my paper Concerning the Separation of Point Sets by Curves.[2]

THEOREM 2. *If a and b are distinct bounded complementary domains of the continuous curve M and the boundaries of a and b have a point P in common then P belongs either to the outer boundary of a or to the outer boundary of b.*

Theorem 2 may be proved with the aid of theorem 1.

THEOREM 3. *If J_1 and J_2 are simple closed curves enclosing the point O and each of the mutually exclusive arcs A_1B_1 and A_2B_2 has one end-point on J_1 and the other one on J_2 but no point, except its end-points, in common either with J_1 or with J_2 then the point set $J_1 + J_2 + A_1B_1 + A_2B_2$ contains a simple closed curve that encloses O and contains either A_1B_1 or A_2B_2.*

Theorem 3 may be proved with the aid of propositions established in my paper "On the Foundations of Plane Analysis Situs."[3]

THEOREM 4. *If D_1 and D_2 are distinct complementary domains of a continuous curve and B_1 and B_2 are their respective boundaries and B is the boundary of a complementary domain of the point set $D_1 + D_2 + B_1 + B_2$ then the three point sets B, B_1 and B_2 do not have more than two points in common.*

THEOREM 5. *If M is a continuous curve and K is the set of all points X such that X is common to the boundaries B_1 and B_2 of two distinct complementary domains D_1 and D_2 of M and is on the boundary of some complementary domain of the point set $D_1 + D_2 + B_1 + B_2$ then K is countable.*

Theorem 5 may be easily established with the aid of theorem 4 and the fact that no continuum has more than a countable number of complementary domains.

THEOREM 6. *If D is a bounded complementary domain of a continuous curve M and J is the outer boundary of D and I is the set of all those points of M that lie within J, then* (1) *no point of I belongs to a simple closed curve that lies in M and encloses a point of J,* (2) *if T is a maximal connected subset of I, J contains only one limit point of T.*

Part (1) of the conclusion of theorem 6 may be proved with the use of certain propositions established in my paper "On the Foundations of Plane Analysis Situs."[3] Part (2) may be proved with the help of similar considerations together with theorem 1 of R. L. Wilder's paper "Concerning Continuous Curves."[4]

THEOREM 7. *If the point O lies in a bounded complementary domain of the continuous curve M and K is the point set obtained by adding together all simple closed curves that lie in M and enclose O, then K is closed.*

Proof. Suppose P is a limit point of K. Then if P does not belong to K it is a limit point either of N or of N_2 where N is the sum of the set G of all simple closed curves that lie in M and enclose both O and P and N_2 is the sum of all those that enclose O but not P. I will assume[5] that P is a limit point of N. If P lies on the boundary of a complementary domain of M it must lie on the outer boundary of that domain, and if the simple closed curve[1] which constitutes this outer boundary should enclose O then P would belong to K. Suppose that P does not belong to the boundary of a complementary domain of M whose outer boundary encloses O. If there exist any complementary domains of M whose boundaries contain P, let H denote the set of all the outer boundaries of such domains, let U denote the sum of the interiors of all the curves of the set H and let T denote the point set consisting of U plus its boundary. The curve M does not[6] have more than a finite number of complementary domains of diameter greater than a given positive positive number. Hence there are not more than a finite number of curves of the set H of diameter greater than such a number. But the point set composed of any such curve plus its interior is itself a continuous curve and T is the sum of all such point sets. It follows, by a theorem of Sierpinski's,[7] that T is a continuous curve. Therefore $M + T$ is a continuous curve. But P does not belong to the boundary of any complementary domain of $M + T$. Hence there exists[8] a sequence α of simple closed curves $j_1, j_2, j_3, \ldots$, all belonging to $M + T$ and enclosing P, such that for every n the diameter of j_n is

less than $1/n$, and j_n encloses j_{n+1} but does not enclose O. Suppose some curve J of the set G contains two distinct points A and B in common with a curve Q of the set H. If P does not belong to J, the point O is enclosed by the simple closed curve L consisting of a certain arc of J together with a certain interval of the arc APB of Q. But L lies wholly in M, contains P and encloses O. Hence, in this case, P belongs to K. Suppose some curve J of the set G contains only one point A in common with one curve Q_1 of the set H and only one point B in common with a certain other curve Q_2 of the set H and suppose that the points A and B are distinct. If AP denotes one of the arcs of Q_1 from A to P, it may be shown that there is one arc of Q_2 from B to P which has no point except P in common with AP. Let BP denote such an arc of Q_2. Let AB denote the sum of the arcs AP and BP. There is an arc of J which together with AB forms a simple closed curve that lies wholly in M, encloses O and contains P. Hence, in this case also, P belongs to K. Suppose now that some curve of the set G has one and only one point E in common with T, and suppose that E is distinct from P. In this case it can easily be shown that there exist other curves of G which pass through E and that indeed P is a limit point of the sum of all such curves. The point E lies on some curve J_E of the set H. Since M is a continuous curve there exists[9] a positive number d such that every point of M whose distance from P is less than d can be joined to P by a simple continuous arc which lies wholly in M and whose diameter is less than e, the distance from P to E. But there exists a curve J of the set G which passes through E and which contains a point X at a distance from P less than d. Suppose J has no point except E in common with T. Let XP denote an arc which is a subset of M and whose diameter is less than e. Let Z denote the last point that XP has in common with J and let Y denote the first point after Z that it has in common with T. Let ZY denote the interval of XP whose end-points are Z and Y. Let J_Y denote a curve of H which contains Y. There exists an arc YE which is a subset of $J_E + J_Y$. Let ZYE denote the arc obtained by adding together the two arcs ZY and YE. There exists on J an arc EWZ such that the simple closed curve formed by the two arcs EWZ and ZYE encloses O. But this curve is a subset of M and it has two points Y and E in common with T. It follows, by a result established above, that P belongs to K. Suppose finally that no curve of the set G has a point in common with the point set T. If T exists, in other words if there exists at least one simple closed curve C belonging to the set H, then there exists an integer n such that j_n contains at least one point within C and there exists a simple closed curve J of the set G such that j_n encloses at least one point of J and it may be shown that the point set $C + j_n + J$ contains a simple closed curve which encloses O and contains P and therefore P belongs to K. If, on the other hand, T does not exist then the following

argument applies. For each n there exists a simple closed curve J_n which belongs to G and contains points within j_{n+1}. There exist two mutually exclusive arcs A_nB_n and C_nD_n lying on J_n and lying, except for their end-points, wholly within j_n and wholly without j_{n+1}, the points A_n and C_n lying on j_n and the points B_n and D_n lying on j_{n+1}. If A_2 is B_1 let X_2 denote A_2 and let Y_2 denote C_2. If A_2 is not B_1 let X_2 and Y_2 denote C_2 and A_2, respectively, or A_2 and C_2, respectively, according as C_2 is or is not on that arc of j_{n+1} which has B_1 and A_2 as end-points and which does not contain D_1. In any case if B_1 is identical with X_2 let B_1X_2 denote the point B_1 and if B_1 is distinct from X_2 let B_1X_2 denote that arc of j_{n+1} which has B_1 and X_2 as end-points and which does not contain D_1. Similarly, if D_1 is Y_2 let D_1Y_2 denote D_1 and if D_1 is not Y_2 let D_1Y_2 denote that arc of j_{n+1} which has D_1 and Y_2 as end-points and which does not contain B_1. Now let X_1, Y_1, Z_1 and W_1 denote the points A_1, C_1, B_1 and D_1, respectively, and let Z_2 and W_2 denote B_2 and D_2, respectively, or D_2 and B_2, respectively, according as X_2 is identical with A_2 or with C_2. Let X_2Z_2 and Y_2W_2 denote the arcs A_2B_2 and C_2D_2 or the arcs C_2D_2 and A_2B_2, respectively, according as X_2 is identical with A_2 or with C_2. This process may be continued. Thus there exist four infinite sequences $X_1Z_1, X_2Z_2, X_3Z_3, \ldots$; $Y_1W_1, Y_2W_2, Y_3W_3, \ldots$; $Z_1X_2, Z_2X_3, Z_3X_4, \ldots$; and $W_1Y_2, W_2Y_3, W_3Y_4, \ldots$ such that, for every n, (*a*) X_n and Y_n lie on j_n, (*b*) X_nZ_n and Y_nW_n lie, except for their end-points, wholly within j_n and wholly without j_{n+1} and, indeed, one of them coincides with A_nB_n and the other one with C_nD_n, (*c*) Z_nX_{n+1} and W_nY_{n+1} lie on j_{n+1} and have no point in common. Let a denote the point set $P + X_1Z_1 + Z_1X_2 + X_2Z_2 + Z_2X_3 + \ldots$ and let b denote the point set $P + Y_1W_1 + W_1Y_2 + Y_2W_2 + W_2Y_3 + \ldots$. It is easy to see that a and b are simple continuous arcs. These arcs belong to M and have P as a common end-point but they have no other point in common. The point set $J_1 + a + b$ contains a simple closed curve which passes through P and encloses O. Hence P belongs to K. Thus the set K contains all of its limit points.

THEOREM 8. *If the point O lies in a bounded complementary domain of the continuous curve M and K is the point set obtained by adding together all simple closed curves that lie in M and enclose O, then K is the sum of a finite number of continuous curves.*

Proof. It has been shown that K is closed. Suppose it is not the sum of a finite number of continuous curves. Then, since every maximal connected subset of K contains a simple closed curve that encloses O and every one of these curves has a diameter greater than the shortest distance from O to K therefore either there is at least one maximal connected subset of K which is not a continuous curve or there exists a positive number such that infinitely many maximal connected subsets of K have diameters greater than that number. In either case there exist[10] two circles k_1 and

k_2 and a countable infinity of continua $M_1,M_2,M_3, \ldots$ such that (1) each of these continua contains at least one point of k_1 and at least one point of k_2 and is a subset of K and of the point set H which is composed of the two circles k_1 and k_2 together with all points that lie between them, (2) no two of these continua have a point in common and, indeed, no one of them is a proper connected subset of any connected point set which is common to K and H, (3) the point O does not belong to H. Let k_3 denote a circle concentric with k_1 and k_2 and lying between them. For each n,M_n has a point P_n in common with k_3 and k contains a simple closed curve J_n which encloses O and contains P_n. Since, for each n,J_n contains a point of k_3 and encloses O, and O does not lie between k_1 and k_2, therefore J_n contains at least one point of k_1 or at least one point of k_2. Let k_4 denote k_1 or k_2 according as there do or do not exist infinitely many distinct values of n such that J_n contains at least one point of k_1. There exists an infinite sequence of positive integers $n_1, n_2, n_3, \ldots$ such that, for each i,J_{n_i} contains, in common with M_{n_i}, an arc t_i which has one end-point on k_3 and the other one on k_4 and which lies, except for its end-points, wholly between k_3 and k_4. The arcs of the sequence $t_1,t_2,t_3, \ldots$ are mutually exclusive and no two of them can be joined by a connected subset of K that lies wholly between k_3 and k_4. But M is uniformly regular.[11] With the help of this fact it can be seen that there exist two positive integers i and j such that t_i and t_j can be joined by two mutually exclusive arcs E_1F_1 and E_2F_2 which are subsets of M and which lie wholly between k_3 and k_4, the points E_1 and E_2 belonging to t_i and the points F_1 and F_2 belonging to t_j. For each n $(n = 1,2)$ let A_n denote the last point that E_nF_n has in common with M_{n_j} and let B_n denote the first point following A_n that it has in common with M_{n_j}. Suppose that, either when $n = 1$ or when $n = 2,A_nB_n$ contains at least one point of J_{n_i} distinct from A_n. Then, for that value of n, let C_n denote the first such point in the order from A_n to B_n on the arc A_nB_n and let L_n denote the last point of J_{n_i}, other than C_n, in the order from E_n to C_n on the interval E_nC_n of the arc E_nF_n. The interval L_nC_n of the arc E_nF_n has its end-points L_n and C_n on J_{n_i} but it has no other point in common with J_{n_i}. Hence the point O is enclosed by a simple closed curve formed by the arc L_nC_n together with one of the arcs of J_{n_i} whose end-points are L_n and C_n. Therefore every point of L_nC_n belongs to K. But this is impossible since L_nC_n lies wholly between k_3 and k_4 and contains a point A_n of M_{n_i} and a point C_n which does not belong to M_{n_i}. Suppose, on the other hand, that J_{n_i} contains no point of A_1B_1 except A_1 and no point of A_2B_2 except A_2. In this case, for each n $(n = 1,2)$, the arc E_nF_n contains an interval L_nC_n which has only the point C_n in common with J_{n_i} and only the point L_n in common with J_{n_i}. By theorem 3 the point set $J_{n_i} +$

$J_{n_j} + L_1C_1 + L_2C_2$ contains a simple closed curve enclosing O and containing either A_1C_1 or A_2C_2. Hence A_1C_1 or A_2C_2 is a subset of K. Hence P_{n_j} can be joined to P_{n_i} by a connected subset of K lying between k_1 and k_2. Thus again we have a contradiction. The truth of theorem 8 is therefore established.

THEOREM 9. *If O is a point lying in a bounded complementary domain of a continuous curve M and K is the point set obtained by adding together all simple closed curves that lie in M and enclose O, then* (1) *the boundary of every complementary domain of a maximal connected subset of K is a simple closed curve and* (2) *if T is a maximal connected subset of M-K, no maximal connected subset of K contains more than one limit point of T.*

Theorem 9 may be easily proved with the help of theorems 8 and 6.

THEOREM 10. *If A and B are distinct points not belonging to the continuous curve M, there exists a simple continuous arc from A to B which does not disconnect M.*

Proof. If A and B belong to the same complementary domain of M they can be joined by an arc lying in that domain and therefore containing no point of M. Suppose that A lies in a bounded complementary domain of M and B lies in the unbounded one.[12] Let K denote the set of points obtained by adding together all simple closed curves that lie in M and enclose A. By theorem 8, K is the sum of a finite number of continuous curves. For each point P of M-K let M_P denote the maximal connected subset of M-K which contains P and let T_P denote the set M_P together with all its limit points. Let H denote the set of all such point sets T_P for all points P of M-K. According to a theorem of W. L. Ayres,[13] if a continuous curve N is a proper subset of a continuous curve M and d is any positive number, there are not more than a finite number of maximal connected subsets of M-N of diameter greater than d. It can be seen that this proposition remains true if, instead of being a single continuous curve, N is the sum of a finite number of continuous curves. Hence there are not more than a finite number of maximal connected subsets of M-K all of diameter greater than the same positive number. It can be seen that if x and y are any two distinct maximal connected subsets of K then not more than a finite number of elements of H have one point in common with x and another point in common with y. Furthermore no maximal connected subset of M-K has limit points in each of three distinct maximal connected subsets of K. It follows, with the help of theorem 9, that the set H can be expressed as the sum of two sets H_1 and H_2, where H_1 consists of all elements h of H such that h has only one point in common with K, and H_2 is a *finite* set consisting of all those elements h of H such that h has just two points in common with K, these points belonging to different maximal connected subsets of K. Let K_1 denote the set of all points

X of K such that X belongs to some point set of the set H_1, but not to any point set of the set H_2. For each point X of K_1 let N_X denote the point set obtained by adding together all point sets of the set H which contain X. Let Q_1 denote the set of all point sets N_X for all points X of K_1. With the help of the fact that there are not more than a finite number of elements of H of diameter greater than any preassigned positive number it is easy to see that every element of Q_1 is a continuous curve and that there are not more than a finite number of these curves of diameter greater than any preassigned positive number. It is also clear that no two elements of Q_1 have a point in common. Let K_2 denote the finite point set consisting of all points X of K such that X belongs to some point set of the set H_2. Let L denote the point set obtained by adding together all point sets h of the set H such that h contains a point of K_2. Let Q_2 denote the set of all maximal connected subsets of L. The set Q_2 cannot contain more elements than K_2 contains points. Hence Q_2 is a finite set. Every element of Q_2 is a continuous curve and no two of them have a point in common. No curve of the set Q_1 has a point in common with any curve of the set Q_2. Let Q denote the set of all curves q such that q belongs either to Q_1 or to Q_2. If a curve of the countable set Q separates A from B it contains[14] a simple closed curve that separates A from B and thus contains uncountably many points of K, which is impossible. Hence no element of Q separates A from B. For each element q of Q let g_q denote either q or the point set obtained by adding to q all its bounded complementary domains, according as q does or does not fail to separate the plane. Let G_1 denote the set of all point sets g_q for all elements q of Q. Let G_2 denote the set of all cut points P of M such that P lies on some simple closed curve which is a subset of M. By a theorem of G. T. Whyburn's,[15] G_2 is a countable set of points. Let G_3 denote the set of all points X of M such that X is common to the boundaries B_1 and B_2 of two distinct complementary domains D_1 and D_2 of M and is on the boundary of some complementary domain of the point set $D_1 + D_2 + B_1 + B_2$. Let G denote the set of all elements g such that g is either a point set of the set G_1 or a point of the set G_2 or of the set G_3. Let U denote the set of all continua x such that x is either an element of G or a point which does not belong to any element of G. The collection U is an upper semi-continuous[16] collection of mutually exclusive continua and every point of the plane belongs to some one of them and no one of them separates the plane. Furthermore, G is a countable subset of U. But if E is a countable set of *points*, every two points not belonging to E can[17] be joined by an arc containing no point of E. It follows[18] that there exists a simple continuous arc t with end-points at A and B and containing no point of any element of the set G. If a point P of the arc t belongs to at least one segment of t whose end-points belong to the boundary of the same complementary domain of M then

(1) there exists a segment A_PB_P which has this property and which contains every other segment that has this property with respect to this particular point P, and (2) since t contains no point of G_3, there exists *only one* domain D_P complementary to M and having both A_P and B_P on its boundary. Let W denote the set of all points P of t for which A_PB_P exists and for which A_PB_P does not belong to D_P. For each point P of W let $A_PZ_PB_P$ denote a definite arc lying wholly in D_P except for its end-points A_P and B_P and let s_P denote the segment $A_PZ_PB_P - (A_P + B_P)$. Let AB denote the arc obtained by adding M-W to the sum of all the segments s_P for all points P of W. The arc AB does not separate M. For suppose it does; then there exist two points X and Y belonging to M and such that every arc M from X to Y contains at least one point of AB. It can be shown that AB contains a point set CD, which is either a single point or an arc with C and D as end-points, such that every arc in M from X to Y contains a point of CD but such that if u is any proper subinterval of CD then there exists, in M, an arc from X to Y that contains no point of u. If CD were a point it would be a cut point of M, contrary to the fact that AB contains no point of G_1 or of G_2. Hence CD is an interval. For each positive integer n, let C_n denote a point of CD such that the diameter of the interval CC_n of CD is less than $1/n$. The curve M contains an arc t_n from X to Y that has no point in common with the interval C_nD of CD. But t_n contains at least one point of CD. Hence it contains a point of the interval CC_n. But every such point is at a distance from C less than $1/n$. Therefore C is a limit point of that maximal connected subset of $(M + CD) - CD$ that contains X. But $M + CD$ is a continuous curve. Hence[4] M contains an arc XC which has no point except C in common with CD. By a similar argument it may be shown that M contains an arc YC that has only C in common with CD and two arcs XD and YD each having only D in common with CD. The point set $XC + XD$ contains an arc CX_1D and $CY + YD$ contains an arc CY_1D. The arcs CX_1D and CY_1D have in common only their end-points C and D and together they form a simple closed curve J. The arc CD lies, except for its end-points, wholly in one of the complementary domains of J. Let I denote this domain and let E denote the other complementary domain of J. There does not exist an arc with end-points at C and D and lying, except for C and D, wholly in $(E + M) - M$. For if there were such an arc CWD then, according to the method given for constructing AB, the arc CWD would be a part of AB which is impossible since CD is a part of AB. With the help of a theorem proved by C. M. Cleveland[8] and the fact that $M + I$ is a regular continuum it follows that M contains an arc X_2Y_2 which lies wholly in E except for its end-points X_2 and Y_2 which lie, respectively, on the segment CX_1D of the arc CX_1D and the segment CX_2D of the arc CX_2D. The point set consisting of the segments CX_1D and CX_2D and the arc X_2Y_2

contains an arc from X_2 to Y_2 that contains no point of CD. Thus the supposition that theorem 10 is false has led to a contradiction.

[1] Moore, R. L., *Math. Zs.*, **15**, 1922 (254–260).

[2] These PROCEEDINGS, **11**, 1925 (469–476).

[3] *Trans. Amer. Math. Soc.*, **17**, 1916 (131–164).

[4] Wilder, R. L., *Fund. Math.*, **7**, 1925 (340–377).

[5] The case where P is a limit point of N_2 may be treated in a similar manner.

[6] Schoenflies, A., *Die Entwicklung der Lehre von den Punktmannigfaltigkeiten*, Leipzig, 1908.

[7] Sierpinski, W., *Fund. Math.*, **1**, 1920 (44–60).

[8] This proposition has been proved by G. T. Whyburn with the use of a theorem proved, at my suggestion, by C. M. Cleveland and stated in *Bull. Amer. Math. Soc.*, **32**, 1926 (311).

[9] Kline, J. R., *Trans. Amer. Math. Soc.*, **21**, 1920 (p. 453, 2nd footnote); Mazurkiewicz, S., *Fund. Math.*, **1**, 1920 (166–209).

[10] Moore, R. L., *Bull. Amer. Math. Soc.*, **29**, 1923 (289–302), § 3.

[11] Hahn, H., *Wien. Ber.*, **123**, 1914 (2433–2489).

[12] Every other case may be reduced to this one by an inversion.

[13] Ayres, W. L., *Bull. Amer. Math. Soc.*, **32**, 1926 (37).

[14] See theorem 7 of my paper cited in footnote 1; also page 475 of the paper cited in footnote 2.

[15] Whyburn, G. T., *Concerning Cut Points and End-Points of Continua*, presented to the American Mathematical Society, June 12, 1926, but not yet published.

[16] Moore, R. L., *Trans. Amer. Math. Soc.*, **27**, 1925 (416–428).

[17] Kline, J. R., *Bull. Amer. Math. Soc.*, **23**, 1917 (290–291).

ON SIMPLY TRANSITIVE PRIMITIVE GROUPS

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Let G be a simply transitive primitive group of degree n. The subgroup G_1 that leaves one letter fixed is intransitive of degree $n - 1$ and is a maximal subgroup of G. Let H be an invariant subgroup of G_1 of degree $n - m(<n - 1)$. Then the largest subgroup of G in which H is invariant is G_1 and H fixes all the letters of a constituent of G_1. From a general theorem on transitive groups[1] it follows that G_1 contains exactly $m - 1$ non-invariant subgroups conjugate to H under G. If B, on the r letters $b, b_1, \ldots$ fixed by H, is a transitive constituent of G_1, there is a corresponding complete set of r conjugate subgroups $H_1, H_1', \ldots$, conjugate under G_1 and conjugate to H under G. To the transitive constituent C, on the r_1 letters $c, c_1, \ldots$ fixed by H, there corresponds the set $H_2, H_2', \ldots$ of r_1 conjugate subgroups of G_1, and so on. Since of all the m conjugates of H under G found in G_1, only H is invariant in G_1, H is a

characteristic subgroup of G_1. The n subgroups $H, \ldots$ constitute a complete set of conjugates under G. Each is a characteristic subgroup of a maximal subgroup of G, and they are, therefore, permuted in the same way by G as are the n maximal subgroups, that is, as G permutes its letters. Obviously the $m - 1$ non-invariant subgroups of G_1 belonging to the set are permuted according to a constituent of G_1 of degree $m - 1$.[2] But it is not known whether or not this constituent coincides with the constituent of G_1 in the letters, $b, b_1, \ldots, c, c_1, \ldots$ fixed by H.[3] At any rate the following theorem can be proved:

THEOREM I. *If all the transitive constituents of H are of the same degree, or if no two (not of maximum degree in H) belong to the same transitive constituent of G_1, every subgroup of G_1 similar to H is transformed into itself by H.*

The above conclusion is entirely equivalent to the statement that the constituent of G_1 according to which the $m - 1$ subgroups $H_1, H_1', \ldots$ are permuted displaces no letter of H.

Let a be the letter of G fixed by G_1. Let $\alpha, \alpha_1, \ldots$ be the letters of one transitive constituent of H, $\beta, \beta_1, \ldots$ of another, $\ldots$, while $\alpha, \beta, \ldots$ are the letters of a transitive constituent of G_1. We impose upon the transitive constituent $\alpha, \alpha_1, \ldots$ of H the condition that no transitive constituent of H is of higher degree.

The permutation $(ab\ldots)\ldots$ of G transforms G_1 into G_2. The characteristic subgroup H of G_1 is transformed by $(ab\ldots)\ldots$ into the characistic subgroup H_b of G_2. Suppose that H_b is not a subgroup of G_1. Then $\{G_1, H_b\} = G$. If H_b fails to connect transitively letters of H and letters $a, b, \ldots$ fixed by H, $\{G_1, H_b\}$ is intransitive. Hence, if H_b is not a subgroup of G_1, one of its permutations unites letters of H and letters fixed by H. If α and x (x is one of the m letters $a, b, \ldots$ fixed by H) are in the same transitive constituent of H_b, so also are the other letters $\alpha_1, \ldots$ of that transitive constituent of H. For H is a subgroup of G_2, and the permutations of H transform H_b into H_b. Then H_b has a transitive constituent $x, \alpha, \alpha_1, \ldots$ of higher degree than any transitive constituent of H, which is absurd. This proves the theorem if all the transitive constituents of H are of the same degree. Now none of the letters $\alpha, \alpha_1, \ldots, \beta, \beta_1, \ldots$ of this one transitive constituent of G_1 are in a transitive constituent of H_b with one of the m letters $a, b, \ldots$. The letters $\lambda, \lambda_1, \ldots$ of a transitive constituent of H of lower degree are by hypothesis the letters of a transitive constituent of G_1. The transitive constituent $x, \lambda, \lambda_1, \ldots$ of H_b contains all the letters of the transitive constituent $\lambda, \lambda_1, \ldots$ of G_1. Then α and λ are in different transitive constituents of $\{G_1, H_b\}$. But $\{G_1, H_b\} = G$. Therefore under the given conditions H_b is a subgroup of G_1, and each of the $m - 1$ non-invariant subgroups $H_1, H_1', \ldots, H_2, \ldots$ of G_1 is transformed into itself by every permutation of H.

THEOREM II. *Let G be a simply transitive primitive group in which each of the m subgroups $H, H_1, \ldots$ of G_1 is transformed into itself by every permutation of H, the characteristic subgroup of G_1. If the degree of the group generated by the complete set of conjugates $H_1, H_1', \ldots$ of G_1 is less than $n - 1$, the letters of G_1 left fixed by it are the letters of one or more of the transitive constituents whose letters are already fixed by H.*

Let the invariant subgroup $K = \{H_1, H_1', \ldots\}$ of G_1 fix letters of G_1. There are conjugate subgroups $K_1, K_1', \ldots$, similar to K, permuted according to a transitive constituent X of G_1. Now K_1 is generated by some of the m subgroups $H, H_1, \ldots$. All the permutations of X except the identity permute subgroups $K_1, K_1', \ldots$. Then no permutation of X, the identity excepted, is in a constituent of H, because every permutation of H transforms each of the subgroups $H_1, H_1', \ldots, H_2, \ldots$ into itself, and therefore each of the subgroups $K_1, K_1', \ldots$ into itself. What is true of the conjugates $K_1, K_1', \ldots$ is true of all such conjugate sets of G_1. Hence the only letters of G_1 fixed by K are letters already fixed by H.

THEOREM III. *If only one transitive constituent of G_1 is an imprimitive group (of order f), G_1 is of order f.*

Let B be the imprimitive constituent of G_1. Suppose that a subgroup H of G_1 corresponds to the identity of B. H displaces all the $n - m$ letters of v primitive constituents of G_1. The $m - 1$ other letters of G_1 are distributed among w transitive constituents $B, C, \ldots$. By theorem I each of the non-invariant subgroups $H_1, H_1', \ldots$ (similar to H) of G_1 is transformed into itself by H. By theorem II, the group $K = \{H_1, H_1', \ldots, H_1^{(r-1)}\}$ displaces all the letters of H as well as the letters of the imprimitive constituent B. If one of the generators $H_1, H_1', \ldots$ of K fixes all the letters of a transitive constituent of G_1, K fixes all the letters of that constituent. Hence H_1, and of course also H, has $v + 1$ or more transitive constituents. But because an invariant subgroup of a primitive group is transitive, H has only v transitive constituents.

COROLLARY I. *If all the transitive constituents of G_1 are primitive groups, G_1 is a simple isomorphism between its transitive constituents.*

Each of the primitive constituents of G_1 may be put in turn in the place of the imprimitive constituent of theorem III.

COROLLARY II. *If G_1 has an intransitive constituent of order f, and if all the transitive constituents on the remaining letters of G_1 are primitive groups, G_1 is of order f.*

[1] Manning, *Bull. Amer. Math. Soc.*, **13**, 20 (1906); *Trans. Amer. Math. Soc.*, **19**, 129 (1918).

[2] Miller, *Proc. London Math. Soc.*, **28**, 535 (1897).

[3] Rietz, *Amer. J. Math.*, **26**, 10, line 23 (1904).

QUADRATIC FORMS WHICH REPRESENT ALL INTEGERS

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We shall give generalizations of the classic theorem that every positive integer is a sum of four squares. We seek all sets of positive integers $a, b, \ldots$ such that every positive integer can be expressed in the form $f = ax^2 + by^2 + \ldots$. We may arrange the terms so that $a \leqq b \leqq c \ldots$ Since f shall represent 1 and 2, evidently $a = 1$, $b < 3$. Hence either $f = x^2 + y^2 + cz^2 + \ldots$ or $f = x^2 + 2y^2 + cz^2 + \ldots$. If $c > 3$ in the former, then $f \neq 3$. If $c > 5$ in the latter, then $f \neq 5$. Hence the sum of the first three terms of f is one of the following ternary forms:

$$t_1 = x^2 + y^2 + z^2, \quad t_2 = x^2 + y^2 + 2z^2, \quad t_3 = x^2 + y^2 + 3z^2,$$
$$t_4 = x^2 + 2y^2 + 2z^2, \quad t_5 = x^2 + 2y^2 + 3z^2, \quad t_6 = x^2 + 2y^2 + 4z^2$$
$$t_7 = x^2 + 2y^2 + 5z^2.$$

The least positive integer l_i not represented by t_i is as follows: $l_1 = 7$, $l_2 = 14$, $l_3 = 6$, $l_4 = 7$, $l_5 = 10$, $l_6 = 14$, $l_7 = 10$. This proves that not all positive integers are represented by any form $ax^2 + by^2 + cz^2$ in which a, b, c are positive integers. It proves also that, if $f = t_i + d_iu^2 + \ldots$ represents all positive integers, then $d_i \leqq l_i$. By hypothesis, d_i is not less than the coefficient of z^2. Hence the first four coefficients of f are those in one of the following 54 sets:

$$\begin{array}{ll} 1,1,1,s\ (s = 1, \ldots, 7); & 1,2,2,s\ (s = 2, \ldots, 7); \\ 1,1,2,s\ (s = 2, \ldots, 14); & 1,2,3,s\ (s = 3, \ldots, 10); \\ 1,1,3,s\ (s = 3,4,5,6); & 1,2,4,s\ (s = 4, \ldots, 14); \\ & 1,2,5,s\ (s = 6, \ldots, 10), \end{array} \qquad (1)$$

or else 1,2,5,5. But the latter are the coefficients of

$$q = x^2 + 2y^2 + 5z^2 + 5u^2,$$

which does not represent 15, since $x^2 + 2y^2$ does not represent 5, 10 or 15, while $z^2 + u^2 \neq 3$. This exceptional value was overlooked by Ramanujan.[1] Without proof he stated empirical theorems on the forms of numbers represented by $t_1, \ldots, t_7$. Recently these theorems have been completely proved by the writer.[2] From them we readily conclude that every positive integer can be represented by each of the 54 forms in four variables whose coefficients are given by (1), and that q represents all positive integers except 15.

Consider a form $f = ax^2 + by^2 + \ldots$ in $n \geqq 5$ variables which represents all positive integers. We have shown that the first four coefficients

are 1,2,5,5 or a set (1). In the latter case the remaining coefficients are arbitrary. The only interesting case is, therefore, $f = q + ev^2 + \ldots$ such that no quaternary form, obtained by deleting all but four terms of f, represents all integers. Then $e < 16$, since otherwise $f = 15$ would require $q = 15$, which was seen to be impossible. The values 6,7,8,9,10 of e are excluded, since the abridged form $x^2 + 2y^2 + 5z^2 + ev^2$ was seen to represent all positive integers. Hence $e = 5,11,12,13,14$ or 15. In these respective cases, we see that f represents 15 when $z = u = v = 1$; $x = 2, v = 1; x = y = v = 1; y = v = 1; x = v = 1; v = 1$; with all further variables zero. We may, therefore, state the

THEOREM. *If, for* $n \geqq 5$, $f = a_1x_1^2 + \ldots + a_nx_n^2$ *represents all positive integers, while no sum of fewer than n terms of f represents all positive integers, then* $n = 5$ *and* $f = x^2 + 2y^2 + 5z^2 + 5u^2 + ev^2$ $(e = 5,11,12,13,14,15)$, *and these six forms f actually have the property stated.*

[1] Ramanujan, *Proc. Cambridge Phil. Soc.*, **19**, 11 (1916–9).

[2] Dickson, *Bull. Amer. Math. Soc.*, **33** (1927).

CONGRUENCES OF PARALLELISM OF A FIELD OF VECTORS

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1. In a geometry of paths the equations of the paths may be written in the form

$$\frac{dx^i}{dt}\left(\frac{d^2xj}{dt^2} + \Gamma^j_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt}\right) - \frac{dx^j}{dt}\left(\frac{d^2x^i}{dt^2} + \Gamma^i_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt}\right) = 0, \quad (1.1)$$

t being a general parameter and Γ^i_{kl} functions of the k's which are symmetric in k and l; a repeated Latin index indicates the same from 1 to n of that index. These functions serve to define infinitesimal parallelism of vectors and accordingly are called the coefficients of the affine connection. We may go further and say that if λ^i are the components of a field of contravariant vectors and C is any curve of the space (at points of which the x's are given as functions of a parameter t), the vectors of the field at points of C are parallel with respect to C, when, and only when,

$$(\lambda^h\lambda^i_{,j} - \lambda^i\lambda^h_{,j})\frac{dxj}{dt} = 0, \quad (1.2)$$

where

$$\lambda^i_{,j} = \frac{\partial\lambda^i}{\partial xj} + \lambda^h\Gamma^i_{hj}. \quad (1.3)$$

These equations are of such a form that if vectors λ^i are parallel with respect to C so also are $\lambda^i\varphi$, where φ is any function of the x's. This is a generalization of the parallelism of Levi-Civita for a Riemannian space.

2. If for a given vector-field λ^i the determinant $|\lambda^i_{,j}|$ is not zero, a necessary and sufficient condition that the determinant $|\bar{\lambda}^i_{,j}|$ for $\bar{\lambda}^i = \varphi\lambda^i$ be zero, that is, that the determinant

$$\left| \varphi\lambda^i_{,j} + \lambda^i \frac{\partial\varphi}{\partial x^j} \right| \tag{2.1}$$

be zero, is that φ be a solution of the equation

$$\begin{vmatrix} \varphi & -\dfrac{\partial\varphi}{\partial x^1} & \cdots\cdots & -\dfrac{\partial\varphi}{\partial x^n} \\ \lambda^1 & \lambda^1_{,1} & \cdots\cdots & \lambda^1_{,n} \\ \lambda^2 & \lambda^2_{,1} & \cdots\cdots & \lambda^2_{,n} \\ \lambda^n & \lambda^n_{,1} & \cdots\cdots & \lambda^n_{,n} \end{vmatrix} = 0. \tag{2.2}$$

Moreover, the rank of (2.2) is $n - 1$ for each solution. Hence, in considering a vector-field λ^i we assume that the components are changed by a factor φ, if necessary, so that the rank of $|\lambda^i_{,j}|$ is at most $n - 1$. We say that then the field is *normal*. This is a generalization of a unit, or null, vector-field in a Riemannian space. For, in this case we have $\lambda_i\lambda^i_{,j} = 0$ and consequently $|\lambda^i_{,j}| = 0$.

If the rank of $|\lambda^i_{,j}|$ is $n - r$, there are r independent vector-fields μ^i_α ($\alpha = 1, \ldots, r$) which satisfy

$$\mu^j\lambda^i_{,j} = 0, \tag{2.3}$$

and the general solution is given by

$$\mu^i = \sum_{\alpha}^{1,\ldots,r} \psi^\alpha \mu^i_\alpha, \tag{2.4}$$

where the ψ's are arbitrary functions of the x's.

When μ^i satisfies (2.3), along each curve of the congruence defined by

$$\frac{dx^1}{\mu^1} = \cdots\cdots = \frac{dx^n}{\mu^n},$$

the vectors λ^i are parallel, as follows from (1.2). Moreover, it follows that the vectors $\varphi\lambda^i$ are parallel also, whatever be φ. Accordingly we say that each solution μ^i of (2.3) defines a *congruence of parallelism* of the field λ^i.

When $|\lambda^i_{,j}|$ is of rank $n - r$, we say that the field λ^i is *general* or *special*, according as the rank of the matrix of the last n rows of (2.2) is $n - r + 1$ or $n - r$. When the field is special, and also when it is general and

$r > 1$, equation (2.2) is satisfied by every function φ. When $r = 1$ and the field is general, equation (2.2) reduces to

$$\mu^j \frac{\partial \varphi}{\partial xj} = 0. \tag{2.5}$$

Suppose that the field is general and that φ is a solution of (2.5) when $r = 1$, or any function whatever when $r > 1$. The equations

$$\mu^j \left(\varphi \lambda^i_{,j} + \lambda^i \frac{\partial \varphi}{\partial xj} \right) = 0 \tag{2.6}$$

are satisfied by all vectors μ^i defined by (2.4) for which the functions ψ^α satisfy the equation

$$\sum_{\alpha}^{1,\ldots,r} \psi^\alpha \mu^j_\alpha \frac{\partial \varphi}{\partial xj} = 0.$$

If there were a solution of (2.6) not expressible in the form (2.4), then from (2.6) we have equations of the form $\lambda^i = \varphi^j \lambda^i_{,j}$, in which case the rank of the matrix of the last n rows of (2.2) is $n - r$. Hence when the field is general, all the solutions of (2.6) are expressible in the form (2.4), that is, on replacing λ^i by $\varphi\lambda^i$ no new congruences of parallelism are obtained.

When the field is special, the determinant (2.1) is of rank $n - r$ at most. Consequently if φ is such that not all of the equations

$$\mu^j_\alpha \frac{\partial \varphi}{\partial xj} = 0 \qquad (\alpha = 1, \ldots, r)$$

are satisfied, there is another solution, say $\mu^i_{\alpha+1}$, of equations (2.6). Evidently it is such that $\mu^j_{\alpha+i} \frac{\partial \varphi}{\partial xj} \neq 0$. If μ^i is any other solution of (2.6) not of the form (2.4), on eliminating λ^i from (2.6) and from the similar equations when μ^j is replaced by $\mu^j_{\alpha+1}$, we have

$$\left(\mu^j_{\alpha+1} \mu^k \frac{\partial \log \varphi}{\partial x^k} - \mu^j \mu^k_{\alpha+1} \frac{\partial \log \varphi}{\partial x^k} \right) \lambda^i_{,j} = 0, \tag{2.7}$$

and consequently μ^i is expressible linearly in terms of μ^i_β ($\beta = 1, \ldots, r + 1$). Hence for the given function φ all solutions of (2.6) are expressible linearly in terms of these $r + 1$ vectors. For another function, say φ_1, there is at most one field $\mu^i_{\alpha+2}$ other than $\mu^i_\alpha (\alpha = 1, \ldots, r)$. But in this case we have the equations obtained from (2.7) on replacing φ in the first term of the left-hand member by φ_1 and μ^j throughout by $\mu^j_{\alpha+2}$. Consequently the change of the function φ does not yield new congruences of parallelism.

Gathering these results together, we have:

When a vector-field λ^i is normal and the rank of $|\lambda^i_{,j}|$ is $n - r$, there are r

independent congruences of parallelism, unless the rank of the matrix of the last n rows of (2.2) *is* $n - r$; *in the latter case there are* $r + 1$ *independent congruences of parallelism; moreover, in either case any linear combination of the vectors defining a congruence of parallelism also defines such a congruence.*

When, in particular, $\lambda^i_{,j} = \lambda^i\sigma_j$, where σ_j is any covariant vector, the vectors λ^i are parallel with respect to any curve in the space, but only in case σ_j is a gradient is parallelism independent of the curve.

3. We consider the converse problem: Given a vector field μ^i to determine the vector fields λ^i of which the former is a congruence of parallelism. We assume that the coördinate system x^i is that for which $\mu^\sigma = 0$ ($\sigma = 2, \ldots, n$).[1] In this coördinate system the equations (2.3) for the determination of the λ's is

$$\frac{\partial\lambda^i}{\partial x^1} + \lambda^j\Gamma^i_{j1} = 0. \tag{3.1}$$

Any set of functions λ^i satisfying these equations are the components in the x's of a vector field with respect to which the congruence μ^i is the congruence of parallelism. A set of solutions is determined by arbitrary values of λ^i for $x^1 = 0$, that is, by n arbitrary functions of $x^2, \ldots, x^n$. In particular, the n sets of solutions λ^i_α, where α for $\alpha = 1, \ldots, n$ determines the set and i the component, determined by the initial values $(\lambda^i_\alpha)_0 = \delta^i_\alpha$ are independent, since the determinant $|\lambda^i_\alpha|$ is not identically zero; here δ^i_α is zero or 1 according as $i \neq \alpha$ or $i = \alpha$. Moreover, from the form of (3.1) it follows that $\lambda^i = \sum_\alpha^{1,\ldots,n} \varphi^\alpha\lambda^i_\alpha$ is also a solution, where the φ's are any functions of $x^2, \ldots, x^n$. Hence we have:

For any congruence μ^i *there exist n independent vector fields* λ^i_α *with respect to which the given congruence is the congruence of parallelism; moreover, the field*

$$\lambda^i = \sum_\alpha^{1,\ldots,n} \varphi^\alpha\lambda^i_\alpha \qquad (i = 1, \ldots, n)$$

possesses the same property, when the φ*'s are any solutions of the equation*

$$\mu^i \frac{\partial\varphi}{\partial x^i} = 0,$$

the coördinates x^i *be any whatever.*

[1] Eisenhart, *Riemannian Geometry*, Princeton Univ. Press, 1926, p. 5.

CONCERNING CERTAIN TYPES OF CONTINUOUS CURVES[1]

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A point set M has property S[2] provided that for every positive number ϵ, M can be expressed as the sum of a finite number of connected point sets each of diameter less than ϵ. A point P of a continuum M is an end-point of M if and only if it is true that P is not a limit point of any connected subset of M minus any subcontinuum of M which contains P. If M is a continuous curve, the point P of M is an end-point provided no arc of M has P as an interior point. I have recently shown[3] that this latter definition for an end-point of a continuous curve is equivalent to the one given by Wilder.[4] The term acyclic continuous curve[5] will be used to designate a continuous curve which contains no simple closed curve.

THEOREM 1. *In order that a bounded continuum M should be a continuous curve every subcontinuum of which is a continuous curve it is necessary and sufficient that every connected open*[6] *subset of M should have property S.*

Proof. The condition is sufficient. Let M denote a bounded continuum every connected open subset of which has property S. Then since M itself is an open subset of M, M has property S, and by a theorem of Sierpinski's[7] it follows that M is a continuous curve. Suppose some subcontinuum N of M is not a continuous curve. Then by a theorem of R. L. Moore's,[8] there exist two concentric circles C_1 and C_2, $C_1 > C_2$, and a countable infinity of subcontinua of N: N^*, N_1,N_2,N_3....., such that (1) each of these continua has at least one point on each of the circles C_1 and C_2 and is a subset of the point set L consisting of the circles C_1 and C_2 together with all those points of the plane which lie between these circles, (2) each of these continua, save possibly N^*, is a maximal connected subset of the set of points common to N and L, and (3) N^* is the sequential limiting set of the sequence of continua N_1,N_2,N_3,..... Let C_3 be a circle concentric with C_1 and C_2 and of radius $1/2$ the sum of the radii of C_1 and C_2. Then it is easily shown that there exists a countable infinity of continua T^*, T_1,T_2,T_3,....., such that if L_0 denotes the set of points consisting of the circles C_1 and C_3 plus all those points of the plane which lie between these two circles, then (1) T^* is a subset of N^* and also of L_0, and for each positive integer n, T_n has at least one point on each of the circles C_1 and C_3 and is a maximal connected subset of the set of points common to N_n and L_0, and (2) T^* is the sequential limiting set of the sequence of continua T_1,T_2,T_3,..... For each positive integer n, let X_n denote a point common to N_n and C_2. The set of points $X_1 + X_2 + X_3 +$ ..... has a limit point A which belongs to both N^* and

C_2. Let ϵ denote the distance between C_2 and C_3. Since M is connected im kleinen, it is easily seen that the set of points $X_1 + X_2 + X_3 + \ldots\ldots$ contains a subsequence $X_{n_1}, X_{n_2}, X_{n_3}, \ldots\ldots$, such that (1) A is the sequential limit point of this sequence, and (2) for every positive integer i, X_{n_i} and A are the extremities of an arc t_i which is a subset of M and is of diameter less than $\epsilon/5i$. Let H denote the set of points $\sum_{i=1}^{\infty} t_i + \sum_{i=1}^{\infty} N_{n_i}$. Clearly H is connected.

Now let K denote the set of all points common to L_0 and N. For every positive integer i, T_{n_i} is a maximal connected subset of the closed point set K. Hence, by a theorem due to Zoretti,[9] there exists, for every i, a simple closed curve J_{n_i} which encloses T_{n_i} and contains no point of K and is such that every point of J_{n_i} is at a distance less than $\epsilon/2i$ from some point of T_{n_i}. Let K_M denote the set of points common to L_0 and M. For each integer $i > 0$, let I_i denote the set of points common to J_{n_i} and K_M. Let W denote the set of points $T^* + I_1 + I_2 + I_3 + \ldots\ldots$ Then W is a closed subset of M. Clearly W has no point in common with H. Let E denote the maximal connected subset of $M - W$ which contains H. Then clearly $M - E$ is closed. Hence E is a connected open subset of M, and by hypothesis must have property S. But since E contains infinitely many continua T_{n_i} all of diameter greater than half the distance from C_1 to C_3 and no two of which can be joined by a connected point set which is common to E and L_0, it easily follows by an argument very similar to that used by R. L. Moore to prove theorem 4 of his paper "Concerning Connectedness im kleinen and a Related Property"[10] that E does not have property S. Thus the supposition that M contains a subcontinuum which is not a continuous curve leads to a contradiction. Hence the condition is sufficient.

The condition is also necessary. Let N denote any definite connected open subset of a continuous curve every subcontinuum of which is a continuous curve, and let ϵ denote any definite positive number. Now unless N itself is of diameter $\leq \epsilon/4$, then N contains two points X_1 and Y_1 whose distance apart is $>\epsilon/4$. By a theorem of R. L. Moore's,[11] N contains an arc t_1 from X_1 to Y_1. Now H. M. Gehman has shown[12] that M cannot contain more than a finite number of mutually exclusive continua each of diameter greater than $\epsilon/4$. It readily follows that $N - t_1$ contains not more than a finite number of maximal connected subsets of diameter greater than $\epsilon/4$. Suppose it contains n such subsets. Let these be ordered $K_1, K_2, K_3, \ldots\ldots K_n$. Then by R. L. Moore's theorem quoted above, for each integer $i \leq n$, K_i contains an arc t_{i+1} of diameter $>\epsilon/4$. By Gehman's theorem it follows that $N - (t_1 + t_2 + \ldots + t_{n+1})$ contains at most a finite number of maximal connected subsets $H_1, H_2, H_3 \ldots\ldots H_m$ each of diameter $> \epsilon/4$. Again, for each integer $i \leq m$,

H_i contains an arc t_{n+1+i} of diameter $>\epsilon/4$. Let this process be continued. Since no two arcs t_i and t_j obtained in this way have a common point, and since each arc is of diameter greater than $\epsilon/4$, it follows by Gehman's theorem that this process must terminate after a finite number of steps. Thus we get a finite collection of arcs $t_1,t_2,t_3,\ldots.t_k$, such that every point of N is at a distance $\leq \epsilon/4$ from some one of these arcs. Let T denote the point set $t_1 + t_2 + t_3 + \ldots. + t_k$. Then since T has property S,[13] T is the sum of a finite number of connected point sets $k_1,k_2,k_3,\ldots.k_n$, each of diameter less than $\epsilon/4$. For each integer $i \leq n$, let l_i denote the set of all those points of N which can be joined to some point of k_i by a connected subset of N of diameter $\leq\epsilon/4$. Then for each integer $i \leq n$, l_i is connected and of diameter less than ϵ, and since every point of N is at a distance $\leq\epsilon/4$ from some set k_i, it follows that $N = l_1 + l_2 + l_3 + \ldots. + l_n$. Hence N has property S, and the theorem is proved.

THEOREM 2. *In order that a bounded continuum M should be an acyclic continuous curve it is necessary and sufficient that every connected subset of M should be uniformly connected im kleinen.*

Proof. The condition is sufficient. Let M denote any bounded continuum every connected subset of which is uniformly connected im kleinen. Evidently M must be a continuous curve. It must also be acyclic. For suppose it contains a simple closed curve J. Then if P denotes a point of J, the set $J - P$ is connected, but clearly it is not uniformly connected im kleinen, contrary to hypothesis. Hence M is an acyclic continuous curve.

The condition is also necessary. Let M denote any acyclic continuous curve, and let N be any connected subset of M. Suppose N is not uniformly connected im kleinen. Then there exists a positive number ϵ such that N contains two infinite sequences of points $X_1,X_2,X_3,\ldots.$, and $Y_1,Y_2,Y_3,\ldots.$, having the property that for each positive integer n, the distance from X_n to Y_n is less than $1/n$, but such that for no integer n can X_n be joined to Y_n by a connected subset of N of diameter less than ϵ. Now since M is uniformly connected im kleinen, there exists an integer k such that X_k and Y_k can be joined by an arc t of M of diameter less than $\epsilon/2$. And since, by a theorem of R. L. Wilder's,[14] every connected subset of an acyclic continuous curve is arcwise connected, N contains an arc t' from X_k to Y_k. But since t' must be of diameter greater than ϵ, t and t' cannot be identical. Hence their sum contains a simple closed curve. But this is contrary to our hypothesis that M is acyclic. Hence the condition of theorem 2 is necessary.

THEOREM 3. *In order that a bounded continuum M should be a continuous curve it is necessary and sufficient that the set of all the non-end-points of M should be uniformly connected im kleinen.*

Proof. The condition is sufficient. Let M denote a bounded con-

tinuum having the property that the set K of all the non-end-points of M is uniformly connected im kleinen. Now every point of $M - K$ is a limit point of K. Hence if K' denotes the set K plus all of its limit points, then $K' = M$. And by a theorem of R. L. Moore's,[15] since K is uniformly connected im kleinen, K' is connected im kleinen. Hence K', or M, is a continuous curve.

The condition is also necessary. Let K denote the set of all the non-end-points of a continuous curve. Since M is uniformly connected im kleinen, for every positive number ϵ there exists a positive number δ_ϵ such that every two points X_1 and X_2 of K whose distance apart is less than δ_ϵ are the extremities of an arc X_1X_2 of M of diameter less than ϵ. But since by definition, no end-point of M is interior to any arc of M, therefore every such arc X_1X_2 must be a subset of K. Hence K is uniformly connected im kleinen, and the theorem is proved.

THEOREM 4. *In order that a bounded continuum M should be a continuous curve it is necessary and sufficient that the set of all the non-end-points of M should have property S.*

Proof. The condition is sufficient. Let M be a bounded continuum having the property that the set K of all the non-end-points of M has property S. Let ϵ denote any definite positive number, and let K be expressed as the sum of a finite number of connected point sets $K_1, K_2, K_3 \ldots K_n$, each of diameter less than ϵ. Since every point of $M - K$ is a limit point of K, it follows that if K_i' denotes the set K_i plus all of its limit points, then $M = K_1' + K_2' + K_3' + \ldots + K_n'$. But for every integer $i \leq n$, K_i' is connected and of diameter less than ϵ. Hence M has property S, and by Sierpinski's theorem,[16] it is a continuous curve.

That the condition of theorem 4 is necessary follows from theorem 3 and from a theorem of R. L. Moore's[17] to the effect that every bounded point set which is uniformly connected im kleinen has property S.

THEOREM 5. *Every strongly[18] connected subset of a continuous curve every subcontinuum of which is a continuous curve is strongly connected im kleinen.*[18]

Proof. Let N denote any strongly connected subset of a continuous curve M every subcontinuum of which is a continuous curve. Suppose N is not strongly connected im kleinen at some one of its points P. Then there exists a circle C having P as center and an infinite sequence of points $X_1, X_2, X_3, \ldots$, all belonging to N and to the interior of C and such that P is the sequential limit point of this sequence but such that no one of these points can be joined to P by a subcontinuum of N which is contained in C plus its interior. Since N is strongly connected, X_1 lies together with P in a subcontinuum K_1 of N. And since every subcontinuum of M is a continuous curve, therefore X_1 and P can be joined by an arc t_1 which is a subset of N. Now t_1 must contain at least one point in the exterior of

C. In the order from X_1 to P on t_1, let Y_1 denote the first point belonging to C, and let a_1 denote the arc X_1Y_1 of t_1. Let X_{n_2} denote the first point of the sequence $X_1,X_2,X_3,\ldots$ which does not belong to a_1. The set N contains an arc t_2 from X_{n_2} to P. In the order from X_{n_2} to P on t_2, let Y_2 denote the first point belonging to the closed set of points $C + a_1$, and let a_2 denote the arc $X_{n_2}Y_2$ of t_2. Let X_{n_3} be the first point after X_{n_2} of the sequence $X_1,X_2,X_3,\ldots$ which does not belong to $a_1 + a_2$. Let a_3 be defined with respect to X_{n_3}, P, and the set $C + a_1 + a_2$ just as a_2 was defined with respect to X_{n_2}, P, and $C + a_1$. This process may be continued indefinitely; and thus we get a sequence of arcs $a_1,a_2,a_3,\ldots$, such that (1) no two of them have an interior point of both in common, (2) each of them is a subset of N and of C plus its interior, (3) each of them contains a point of the sequence of points $X_1,X_2,X_3,\ldots$, and (4) P belongs to their limiting set. Since every subcontinuum of M is a continuous curve, it follows from property (1) of this sequence and from a theorem of H. M. Gehman's[12] that for any $\epsilon > 0$, there are not more than a finite number of arcs of this sequence of diameter greater than ϵ. Hence no point other than P, which does not belong to one of the arcs of this sequence, can belong to their limiting set. Therefore, $P + a_1 + a_2 + a_3 + \ldots$. is a closed point set. Not more than a finite number of arcs of the sequence $a_1,a_2,a_3, \ldots$. can have points on C. Let $a_{k_1}, a_{k_2}, \ldots a_{k_n}$ be the ones that do. For some integer $i \leq n$, a_{k_i} plus some infinite subcollection a_{n_1},a_{n_2},a_{n_3}, ... of the above sequence forms a connected point set. Let K denote this connected point set. Then since P is a limit point of the sequence of arcs $a_{k_i},a_{n_1},a_{n_2},a_{n_3},\ldots$, therefore, P is a limit point of K, and the set of points $K + P$ is connected and closed. But $K + P$ is a subset of N and of C plus its interior, and it contains P and a point of the sequence of points $X_1,X_2,\ldots$, contrary to our supposition. Thus the supposition that N is not strongly connected im kleinen leads to a contradiction.

THEOREM 6. *In order that the set of all the non-end-points of a continuous curve M should be a subset of the sum of the boundaries of the complementary domains of M it is sufficient that if A and B are any two points belonging to the same simple closed curve of M, then $M - (A + B)$ is not connected.*

Proof. Let E denote the set of all the non-end-points of a continuous curve M which satisfies the hypothesis of theorem 6. I have recently shown[3] that every point of E either is a cut point of M or else belongs to some simple closed curve of M. Let A denote any point of E. If A is a cut point of M, then by a theorem of R. L. Moore's,[19] A belongs to the boundary of some complementary domain of M. If A is not a cut point of M, then it belongs to some simple closed curve J of M. Now J contains at most a countable number of cut points of M.[20] Hence there exists, on J, a point B which is distinct from A and which is not a cut point of M. The set $M - B$ is connected. But by hypothesis $M - (A + B)$

is not connected. Hence A is a cut point of the set $M - B$. Let $(M - B) - A$ be expressed as the sum of two mutually exclusive sets S_1 and S_2 neither of which contains a limit point of the other. The sets $S_1 + A$ and $S_2 + A$ are connected. Now suppose, contrary to this theorem, that A does not belong to the boundary of any complementary domain of M. Then by a theorem of mine,[3] if X and Y denote points of S_1 and S_2, respectively, M contains a simple closed curve C which encloses A but which neither contains nor encloses any one of the points B, X and Y. Since $S_1 + A$ and $S_2 + A$ are connected sets, each containing a point within C and also a point without C, there must exist points P and Q, common to $S_1 + A$ and P and to $S_2 + A$ and C, respectively. Since C encloses A, P and Q must belong to S_1 and S_2, respectively. But C is a connected subset of $(M - B) - A$, and hence S_1 and S_2 are not mutually separated, contrary to hypothesis. Thus the supposition that A does not belong to the boundary of a complementary domain of M leads to a contradiction. Hence, in any case, A belongs to the boundary of some complementary domain of M; and since A is any point of E, then E must be a subset of the sum of the boundaries of the complementary domains of M.

Examples can easily be constructed to show that the condition of theorem 6 is not necessary.

[1] Presented to the American Mathematical Society, Sept. 9, 1926.

[2] Cf. Moore, R. L., "Concerning Connectedness im kleinen and a Related Property," *Fund. Math.*, **3**, 1921 (233–237).

[3] This theorem is in my paper, "Concerning Continua in the Plane," which has been submitted for publication in the *Trans. Amer. Math. Soc.*

[4] Wilder, R. L., "Concerning Continuous Curves," *Fund. Math.*, **7**, 1925 (358).

[5] This term has recently been introduced by Dr. H. M. Gehman.

[6] An open subset of a closed set M is any subset N of M such that $M - N$ is either vacuous or closed.

[7] "Sur une condition pour qu'un continu soit une courbe jordanienne," *Fund. Math.*, **1**, 1920 (44–66).

[8] "A Report on Continuous Curves from the Viewpoint of Analysis Situs," *Bull. Amer. Math. Society*, **29**, 1923 (296–297).

[9] Zoretti, L., "Sur les fonctions analytiques uniformes," *J. Math. pures appl.*, **1**, 1905 (9–11).

[10] Loc. cit.

[11] "Concerning Continuous Curves in the Plane," *Math. Zeit.*, **15**, 1922 (254–260).

[12] "Concerning the Subsets of a Plane Continuous Curve," *Annals of Math.*, **27**, 1925 (29–46).

[13] That T has property S is a direct consequence of Sierpinski's theorem mentioned in ref. 7.

[14] Loc. cit., p. 377, theorem 20.

[15] "Concerning Connectedness im kleinen and a Related Property," loc. cit., p. 233, theorem 1.

[16] Loc. cit.

[17] Moore, R. L., loc. cit., p. 234, theorem 3.

[18] A point set M is strongly connected provided that every two points of M lie to-

gether in a subcontinuum of M. A point set M is strongly connected im kleinen if for every point P of M and for every positive number ϵ there exists a positive number $\delta_{\epsilon P}$ such that every point X of M whose distance from P is less than $\delta_{\epsilon P}$ lies together with P in a *subcontinuum* of M of diameter less than ϵ. This definition for the term "strongly connected im kleinen" is identical with the one originally given by Hans Hahn [cf. *Weiner Akademie Sitzungsberichte*, **123**, Abt. IIa (2433–2489)], for the term "connected im kleinen." It has been customary, however, by R. L. Moore and others, to use the term "connected im kleinen" in the sense as given by the above definition with the words *connected subset* substituted for the word *subcontinuum*.

[19] "Concerning the Common Boundary of Two Domains," *Fund. Math.*, **6**, 1924 (211).

[20] Cf. Moore, R. L., "Concerning the Cut Points of Continuous Curves and of Other Closed and Connected Point Sets," these PROCEEDINGS, **9**, 1923 (101–106).

SUMMARY OF RESULTS AND PROOFS CONCERNING FERMAT'S LAST THEOREM (SECOND NOTE)

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As in the first note (these PROCEEDINGS, **12**, 1926, 106–9), we shall divide the discussion into two cases. If in the relation

$$x^p + y^p + z^p = 0, \tag{1}$$

with x, y, z, prime to each other, xyz is prime to the odd prime p, this will be referred to as case I, and if one of the integers x, y, z, is divisible by p, this will be called case II.

The theorems I–III, together with corollaries I and II to theorem II, of the first note (using the same notation), can be made much stronger by noting that we have from

$$\begin{aligned} &x + \beta^a y \equiv 0 \pmod{\mathfrak{p}}, \qquad x^p \equiv -\beta^{ap} y^p \pmod{\mathfrak{p}}, \\ &x^p = -y^p - z^p, \qquad y^p + z^p \equiv \beta^{ap} y^p \pmod{\mathfrak{p}}, \\ &y^p(\beta^{ap} - 1) \equiv z^p \pmod{\mathfrak{p}}. \end{aligned} \tag{2}$$

If r is a primitive root of n and we assume $n - 1 \not\equiv 0$ modulo p, the above congruence gives, observing that $\beta \equiv r \pmod{\mathfrak{p}}$,

$$y^p(r^{ap} - 1) \equiv z^p \pmod{\mathfrak{p}}.$$

Since $n - 1 \not\equiv 0$ modulo p, then there exists an unique integer $k < n - 1$ such that

$$r^{ap} - 1 \equiv r^k \pmod{n},$$

and, therefore,

$$\beta^{ap} - 1 \equiv \beta^{jp} \pmod{\mathfrak{p}},$$

where

$$jp \equiv k \pmod{\overline{n-1}}.$$

Hence from (2)

$$y^p \beta^{jp} \equiv z^p \pmod{\mathfrak{p}}.$$

Let s be a rational integer such that

$$ps \equiv 1 \pmod{\overline{n-1}}.$$

Then, raising each side of the last congruence to the power s, we have

$$\beta^j y \equiv z \pmod{\mathfrak{p}},$$

where j depends entirely on a. We may obtain a similar relation involving x and z, therefore, in theorem I, if the relations hold for a particular a, there are, in general, five other values of a for which the relations are satisfied. Corollary II can also be put in the form involving the product of only $n - 2$ factors instead of $(n - 2)^2$.

For a particular n, if the relations in any of the theorems are satisfied for $a_1, a_2, \ldots, a_i$, that is, for particular a's, it follows that

$$x + \beta^{ac} y \equiv 0, \pmod{\mathfrak{p}},$$

where c is one of the values $1, 2, \ldots, i$. This statement furnishes a limitation on the value of x/y modulo n.

The methods of the first note yield easily the following theorem, which is a generalization of Furtwängler's:[2]

THEOREM A. *If $x^p + y^p + z^p = 0$ is satisfied in integers none zero and all prime to the odd prime p, and n is a rational integer dividing $x^k + y^k$, then*

$$q(n) \prod_{a=1}^{k-1} ((1 - v) \operatorname{ind} (\alpha\gamma^a - 1) - q(n)) \equiv 0 \pmod{\mathfrak{p}},$$

where n and k are prime to p and

$$\left\{\frac{\alpha\gamma^a - 1}{\mathfrak{q}}\right\} = \alpha^i, \qquad i = \operatorname{ind}(\alpha\gamma^a - 1),$$

$\mathfrak{q}$ a prime ideal divisor of n in the field $\Omega(\gamma)$, $\alpha = e^{2i\pi/p}$, $\gamma = e^{2i\pi/k}$, v being any one of the values $t, \frac{1}{t}, 1 - t, \frac{1}{1-t}, \frac{t-1}{t}, \frac{t}{t-1}; -\frac{x}{y} = t$.

We also may consider:

THEOREM B. *If the class number of the field defined by a primitive pth root of unity is divisible by p but not by p^2, then $x^p + y^p + z^p = 0$ is impossible*

in integers x, y, z in the field $\Omega(\alpha + \alpha^{-1})$ *prime to each other, none zero, p being an odd prime.* (Cf. Vandiver.[3])

The proof depends on modifications and extensions of Kummer's[4] argument in his memoir of 1857. In two papers by the writer[5] it is shown that Kummer's work was insufficient and inaccurate in several respects, and proofs of two of the four theorems used by Kummer were deduced by the former.[6]

Kummer's methods may be considerably simplified, but his paper will be followed as closely as possible for historical reasons, and hence I shall prove first only a result which is equivalent to that which he attempted to prove, that is:

THEOREM C. *Under the assumptions:* (1) *The class number of the field* $k(\alpha)$, $\alpha = e^{2i\pi/\lambda}$, λ, *an odd prime, is divisible by* λ *but not by* λ^2. (2) *If* $B_\nu \equiv 0 \pmod{\lambda}$, $\nu < (\lambda - 1)/2$, *and* $B_{\nu\lambda} \not\equiv 0 \pmod{\lambda^3}$; *the relation* $x^\lambda + y^\lambda + z^\lambda = 0$ *is not satisfied in rational integers, none zero.*

Consider Kummer's theorems II and III. (Cf. Vandiver,[5] p. 268.) Using his notation, take Fermat's relation in the form $x^\lambda + y^\lambda + z^\lambda = 0$, where λ is an odd prime; the theorems are:

Theorem II: If the class number of $k(\alpha)$ where $\alpha = e^{2i\pi/\lambda}$ is divisible by λ but not by λ^2, and if $B_\nu \equiv 0 \pmod{\lambda}$ but $B_{\nu\lambda} \not\equiv 0 \pmod{\lambda^2}$, then a unit in $k(\alpha)$, which is congruent, modulo λ^2, to a rational integer, is the λth power of a unit in $k(\alpha)$.

Theorem III: If the class number of k is divisible by λ but not by λ^2 and if $F(\alpha)$ is an integer in the field $K(\alpha + \alpha^{-1})$ and also

$$\frac{d_0^{2\nu} 1\, F(e^v)}{dv^{2\nu}} \equiv 0 \pmod{\lambda},$$

then a unit $E(\alpha)$ in k can be found such that

$$E(\alpha)\ F(\alpha) \equiv a \pmod{\lambda},$$

where a is a rational integer.

As to the first theorem, it follows from the assumption as to the class number that any unit in the field may be written in the form

$$\alpha^l E_1^{n_1} E_2^{n_2} \ldots E_{\mu-1}^{n_{\mu-1}} = E, \tag{3}$$

where $\mu = (\lambda - 1)/2$, and

$$E^n(\alpha) = E_n = e(\alpha)\ e(\alpha^\gamma)^{\gamma^{-2n}}\ e(\alpha\gamma^2)^{\gamma^{-4n}} \ldots e(\alpha^{\gamma^{\mu-1}})^{\gamma^{-2(\mu-1)n}},$$

γ a primitive root of λ and

$$e(\alpha) = \sqrt{\frac{(1 - \alpha^\gamma)(1 - \alpha^{-\gamma})}{(1 - \alpha)(1 - \alpha^{-1})}},$$

the n's being rational fractions whose denominators may be taken prime to λ.

Assume then

$$E \equiv a \pmod{\lambda^2},$$

where a is a rational integer, and E is a unit in the field. Setting e^v in place of α and adding the proper multiple of

$$(e^{v\lambda} - 1)/(e^v - 1)$$

to make the result an identity in e^v, then taking logarithms of both members and differentiating i times, $i = 1, 2, \ldots, \lambda - 2$, we obtain, after dividing out factors prime to λ,

$$B_t n_t \equiv 0 \pmod{\lambda},$$

$$t = 1, 2, \ldots, \mu - 1.$$

Hence all the n's are divisible by λ except B_ν. Write then

$$E = E_\nu^{n_\nu} \epsilon^\lambda,$$

where ϵ is a unit in the field.

In order to proceed further, it is necessary to note that Kummer[3] (p. 71) applies theorem II to the unit $\mathfrak{e}_s(\alpha)/\mathfrak{e}_r(\alpha)$ which occurs in the relation

$$T_r(\alpha)^\lambda \equiv \frac{\mathfrak{e}_s(\alpha)}{\mathfrak{e}_r(\alpha)} T_s(\alpha)^\lambda, \pmod{\lambda^2},$$

where the T's are integers in the field $\Omega(\alpha)$ and

$$T_r(\alpha) \equiv a \pmod{\lambda},$$

where a is a rational integer, $\not\equiv 0 \pmod{\lambda}$, and

$$T_s(\alpha) \equiv b \pmod{\lambda}$$

where b is a rational integer, $\not\equiv 0 \pmod{\lambda}$, hence

$$a^\lambda \equiv \frac{\mathfrak{e}_s(\alpha)}{\mathfrak{e}_r(\alpha)} b^\lambda, \pmod{\lambda^2}.$$

From this it is seen that the unit in question is congruent to the λth power of a rational integer modulo λ^2, hence we may write, where c is a rational integer, $\not\equiv 0 \pmod{\lambda}$

$$(E_\nu(\alpha)^{n_\nu} \epsilon^\lambda \equiv c^\lambda \pmod{\lambda^2},$$

and this gives

$$[E_\nu(e^v)]^{n_\nu} \epsilon(e^v)^\lambda = c^\lambda + \lambda^2 \Psi(e^v) + V(e^v)(e^{v\lambda} - 1) + d \cdot \frac{e^{v\lambda} - 1}{e^v - 1}, \quad (4)$$

where d is a rational integer and $\Psi(e^v)$ and also $V(e^v)$ are polynomials in

ϵ^v with rational integral coefficients. Raise both members of this equation to the power $\lambda - 1$ and set $v = 0$, and we obtain

$$\gamma^{n_\nu k(\lambda-1)} (\epsilon(1))^{\lambda(\lambda-1)} \equiv c^{\lambda(\lambda-1)} - d\lambda c^{\lambda(\lambda-2)} \mod \lambda^2,$$

where

$$k = \frac{1 - \gamma^{-n(\lambda-1)}}{1 - \gamma^{-2n}}$$

and, therefore, since $k \equiv 0 \pmod{\lambda}$, and $i^{\lambda(\lambda-1)} \equiv 1 \pmod{\lambda^2}$, for i any integer prime to λ,

$$d \equiv 0 \pmod{\lambda}.$$

Using this value of d in (4), taking logarithms, differentiating $2\lambda\nu$ times and setting $v = 0$ in the result and reducing modulo λ^2, gives, after dividing out factors $\not\equiv 0 \pmod{\lambda}$,

$$n_\nu B_{\nu\lambda} \equiv 0 \pmod{\lambda^3},$$

and by assumption 2 of theorem C we have

$$n_\nu \equiv 0 \pmod{\lambda^3},$$

and this proves Kummer's theorem II if the unit is congruent to the λth power of a rational integer modulo λ^2, and this is the particular case, which, as we have noted, is used in connection with the last theorem.

As shown by Bernstein,[7] theorem III is not really necessary in carrying out Kummer's argument. The theorem, however, is true, but in view of Bernstein's work, I shall not give a proof here. Kummer's assumption II is equivalent to the statement that the second factor of the class number is prime to λ because of the theorem on page 426 of Hilbert.[8]

We have thus proved theorems I, a special case of theorem II and IV of Kummer, and in addition noted that theorem III was unnecessary, hence the theorem C is proved, if we use Dedekind ideals instead of Kummer's and if we note that the case where x, y, z, are each prime to λ is covered by Kummer,[4] pp. 63-5.

We shall now give the outlines of a proof of the

THEOREM D. *Under the assumptions*

1. *None of the Bernoulli numbers*

$$B_k, \quad k = (s\lambda + 1)/2 \qquad (s = 1, 3, \ldots, \lambda - 4)$$

is divisible by λ^2;

2. *The second factor of the class number of the field is prime to* λ,

then

$$x^\lambda + y^\lambda + z^\lambda = 0$$

is not satisfied in rational integers, x, y, z, none zero, if λ *is an odd prime.*

This result was enunciated by the writer[9] without proof. The proof had been obtained about the year 1918 and depended on certain theorems concerning the composition of classes in the field $\Omega(\alpha)$, in particular,

the theorem that if the field contained an ideal belonging to the exponent λ^2, it is necessary that one of the Bernoulli numbers

$$B_k, \quad k = (s\lambda + 1)/2 \qquad (s = 1, 3, \ldots, \lambda - 4)$$

is divisible by λ^2. By a method quite different, Pollaczek[10] obtained a proof of this result and in addition established a number of other results along the same lines.

Under assumptions 1 and 2 of the present theorem, it follows that no ideal in the field exists belonging to the exponent λ^2 and according to the relations given by Bernstein[7] (pp. 514–515), we have the result (which may be proved independently) that a primary number in the field $\Omega(\alpha)$, which is the λth power of an ideal in this field may be expressed as the λth power of a number in the field multiplied by a primary unit. Also, under assumptions 1 and 2 of this theorem, it may be shown that a unit in $\Omega(\alpha)$ which is congruent to a^λ modulo λ^2, where a is a rational integer is the λth power of a unit.

Using then Kummer's argument or Bernstein's[7] modification of it, we may establish our theorem under the assumption that one of the integers x, y, z, has the factor λ.

In another paper the writer[11] proved that if $x^p + y^p + z^p = 0$ is satisfied in integers x, y, z, prime to the odd prime p, then $B_s \not\equiv 0 \pmod{p^2}$, $s = (lp + 1)/2$, $l = p - 4, p - 6, p - 8, p - 10$.
This result taken with what precedes proves theorem D.

Also theorem B follows as a special case of D for x, y and z rational integers and if it is not difficult to extend this result to the case x, y and z integers in the field $K(\alpha + \alpha^{-1})$. Cf. Furtwängler.[12] Full details will follow in articles to appear later.

Kummer[4] (p. 73) verified that his assumptions 2 and 3 held for $\lambda = 37$, 59 and 67. These computations, if correct, show that the second factor of the class number for these particular values of λ is prime to λ and using in addition Kummer's result concerning the last theorem for regular primes λ, it would follow that the last theorem is established for all prime exponents $\lambda < 100$.

[1] Presented to the American Mathematical Society, New York, April, 1926.
[2] *Wiener Bericht*, Abt. IIa, **121**, 1912, 589–592.
[3] *Bull. Amer. Math. Soc.*, **30**, 1924, 229.
[4] *Berlin Math. Abhandlungen*, 1857, 40–74.
[5] *Proc. Nat. Acad. Sci.*, **6**, 1920, 266–269; 416–421.
[6] *Bull. Amer. Math. Soc.*, **28**, 1922, 400–407.
[7] *Nach. Gott., Math.-Physik Klasse*, 1910, 512.
[8] Über die Theorie der algebraischen Zahlkörper, Bericht, 426.
[9] *Bull. Amer. Math. Soc.*, **28**, 1922, 260.
[10] *Math. Zeit.*, **21**, 1924, 24–26.
[11] *Ann. Math.*, **26**, 1925, 217–232.
[12] *Nach. Gott., Math.-Physik Klasse*, 1910, 562.

ERRATA

In the paper by H. W. Rand, J. F. Bovard and D. E. Minnich (these PROCEEDINGS, **12**, no. 9, September, 1926), on page 570, line 4, "adsorption" should read "absorption."

In the paper by Herbert W. Rand and Mildred Ellis (these PROCEEDINGS, **12**, no. 9, September, 1926), on page 572, second line from bottom, *surface* should read *surfaces*.

AN ACKNOWLEDGMENT

The formula for the distribution of the correlation ratio calculated from random data obtained in my paper in these PROCEEDINGS, **11**, no. 10 (October, 1925), pp. 657–662, is equivalent to a result contained in a paper by R. A. Fisher on "The Goodness of Fit of Regression Formulae and the Distribution of Regression Coefficients," *J. Roy. Statist. Soc.*, **85** (1922), pp. 597–612, a fact which unfortunately I overlooked. The credit for the formula belongs to Mr. Fisher.

HAROLD HOTELLING

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